

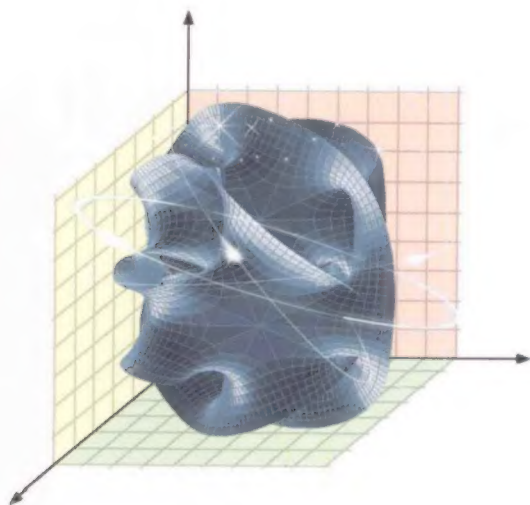


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刘 进 李海峰 刘 煜  
徐培德 贺 川 伍国华 著

On the Gap Phenomenon of  
Curvature Normal for Submanifold



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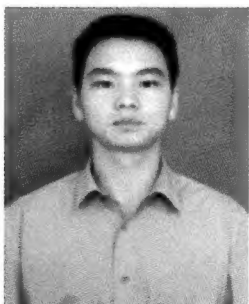
刘煜,湖南岳阳人,1983年生。2010年博士毕业于英国东安格利亚大学。2011年7月起在国防科技大学信息系统与管理学院工作。主要研究领域为编码孔径技术、人脸识别和人脸表情识别、计算机力反馈研究和计算几何。发表SCI论文10余篇,出版专著1部。



徐培德,上海人,1958年生。1987年毕业于国防科技大学系统工程与数学系应用数学专业,获硕士学位。一直从事军事运筹、计算机仿真模拟、图形处理和计算几何等方面的研究。现为国防科技大学教授,中国运筹学会常务理事、副秘书长,全军军事运筹学学会副理事长,中国系统工程学会军事系统工程专业委员会委员,中国计算机模拟学会常务理事,湖南省运筹学会常务理事,《运筹与管理》分区主编。参加和主持科研项目30余项,获省部(军队)级科技进步二等奖4项、三等奖4项,发表学术论文80多篇,已出版教材、专著6部(第一作者4部)。



贺川,男,汉族,四川都江堰人,1985年出生。2003-2010年于中国人民解放军郑州防空兵学院学习,先后获得指挥自动化工程专业学士和军事运筹学硕士学位。2010~2013年于国防科学技术大学信息系统与管理学院学习,获军队指挥学博士学位。主要研究方向为军事运筹分析和航天装备军事应用以及图形处理。



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# 内容简介

Introduction

微分几何之中著名的 Simons - Lawson - Chern - do Carmo - Kobayashi 定理开辟了极小子流形曲率模长间隙现象的课题方向。本书系统地用变分理论对一类与子流形曲率模长有关的泛函进行了研究。全书分为三部分。第 1 部分为第 1 章,介绍曲率模长泛函的研究背景和当前的研究现状。第 2 部分为 3、4、5、6 等 4 章,介绍和推导了本书的理论基础。第三部分为 6、7、8、9、10、11 等 6 章,具体且精细地研究了各种类型的曲率模长泛函,构造了多种例子,讨论了曲率模长泛函的稳定性,特别研究了曲率模长泛函临界点的间隙现象,是本书的核心部分。全书论述严密精炼,适合数学与图形处理专业的研究生及科研工作者参考。

# 前言

Foreword

1965—1970 年左右,五位数学大师发表的三篇著名的文章 (Simons 的《Minimal Varieties in Riemannian Manifolds》,Lawson 的《Local Rigidity Theorems for Minimal Hypersurfaces》和陈省身、do Carmo、Kobayashi 联合发表的《Minimal Submanifolds of a Sphere with Second Fundamental Form of Constant Length》)开辟了子流形几何之中非常奇特的间隙现象研究,引出了等参超曲面数十年悬而未决的系列猜想。

一个自然的问题是,间隙现象是否严格依赖泛函的表达形式?答案是否定的,大量的研究表明,间隙现象是一个较为普遍的现象,在各种具体的泛函之中都有表现。第二个自然的问题是,除了所涉及的具体泛函,抽象形式的泛函在多大范围之内可以推导出间隙现象?这就是本书所要部分回答的问题。

本书以变分法作为主要工具。用变分法研究曲率模长泛函是一个自然的选择。变分法的重要性基于三个主要理由。理由一:在自然科学中的成功应用,特别是在图形处理的新理论、新方法、新算法设计中的作用;理由二:古典曲面论启发对空间形态的认识取决于两个因素:内蕴因素与外蕴因素,即曲面在三维空间的形态不仅由内蕴度量性质的第一基本型决定,也由曲面的嵌入方式即外蕴度量性质的第二基本型决定;理由三:数学家和诺贝尔经济学奖获得者 Nash 发明的 Nash 嵌入定理使任何一个完备的黎曼流形都可嵌入为欧氏空间的子流形,在此意义上,黎曼流形内蕴的研究本质上可以归结于子流形内蕴和外蕴性质的研究。

变分理论的研究历史悠久,最速下降线或者测地线的变分法研究即是子流形变分法的雏形。数学家欧拉、高斯、黎曼、嘉当、陈省身、丘成桐等对变分法做出了重要贡献。一个著名的例子是陈



省身与合作者利用变分法和结构方程讨论了单位球面中极小子流形的某些几何量的间隙现象,并且确定了间隙端点所对应的特殊子流形,是子流形几何中的具有显著地位的定理。以此结论出发,众多数学家进行推广和深入探讨,形成目前子流形几何一个较为重要的方向:间隙现象的研究。

本书的主要目的在于系统地研究和介绍曲率模长泛函的变分理论。全书共分11章,可以归纳为3部分。第1部分为第1章,主要介绍曲率模长泛函的研究历程和国内外研究现状,使读者对于此类泛函有整体的把握。第二部分是基础理论篇,包括2、3、4、5等4章。各章的目的和作用不一。第2章精炼介绍微分几何的基本方程和定理,为后面各章提供预备知识;第3章推导子流形几何的基本方程和变分法基本公式,是本书的理论基础;变分法的计算通常非常复杂,为了简化公式和计算过程,第4章研究子流形第二基本型的组合构造方法:牛顿变换法,并推导了新构造的张量的基本性质,是对第3章内容的扩充和精细;自伴算子是子流间隙现象研究的有效工具,第五章利用牛顿张量设计了多种几何意义明确的自伴算子,并对几种典型函数做了精细的计算。第3部分是本书的核心篇章,包括6、7、8、9、10、11等6章。各章的主题不一。第6章回顾了经典的体积泛函的定义,并陈述了极小子流形曲率模长间隙现象,由此出发,作者定义了最一般的曲率模长泛函。第7章计算了曲率模长泛函的第一变分公式,这是整个研究的基础。在第一变分公式的基础上,在第8章,作者综合运用代数、方程的手段构造了典型的临界子流形。在第9章,作者计算了曲率模长泛函的第二变分公式,适用范围广泛。第10章推导了子流形几何中非常奇异的Simons类型积分不等式。第11章详细讨论各类型曲率模长泛函的间隙现象,定出了间隙端点对应的特殊子流形,是全书的精华部分。

全书由6位作者合作完成。第1、10、11章由刘进完成,第4、9章由李海峰完成,第2、3章由刘煜完成,第5、6章由徐培德完成,贺川、伍国华分别完成第7、8章,全书由刘进统稿。

本书是作者对曲率模长泛函变分问题的一个粗浅阐述,由于作者水平有限,请各位专家不吝赐教。

本书的研究课题受到如下项目资助:国家自然科学基金项目(41001220和51178193),中国博士后科学基金面上资助项目(2012M511411),中国博士后科学基金特别资助项目(2013T60779),国家高技术研究发展计划(863计划)项目:(2012AA121301)。

刘进 李海峰 刘煜 徐培德 贺川 伍国华 等  
2013年6月

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# 第1章 间隙现象的研究概述

中国学者研究子流形一般遵循三篇基本的文献,这三篇文献构成了极小子流形研究方面的“圣经”级参考资料。第一篇基本文献为 Simons 的文章<sup>[1]</sup>,在文中 Simons 利用向量场标架法研究了极小子流形的变分理论和间隙现象;第二篇基本文献为 Chern、do Carmo、Kobayashi 三人合作的文章<sup>[2]</sup>,在文中三位作者利用活动标架法和李代数理论决定出了极小子流形 Simons 间隙现象端点对应的特殊子流形;第三篇文献为陈省身先生在第一篇和第二篇文献的基础上总结而来的小册子,数学史上称为 Kansas 讲义<sup>[3]</sup>,陈省身先生利用活动标架法对<sup>[1,2]</sup>之中的内容进行了简化和深化。

在讲义<sup>[3]</sup>之中,陈先生推导出了子流形的基本结构方程;计算了体积泛函的第一变分;对欧式空间的极小子流形进行了微分刻画,推导出了著名的函数图极小方程,介绍了 Bernstein 定理的演化进程;用复变函数(等温坐标,黎曼曲面)对欧式空间的可定向 2 维极小曲面进行了刻画;对单位球面之中的极小子流形的结构方程进行了推导,给出了几个典型例子,特别是 Clifford 超曲面与 Veronese 曲面;计算了第二基本型的 Laplacian;利用此计算结合精巧的不等式分析推导出了 Simons 积分不等式,利用结构方程和 Frobenius 定理决定出了间隙端点对应的特殊子流形;在第一变分公式的基础上计算了第二变分公式;最后讨论了锥子流形的稳定性的特征值刻画问题。

必须指出,Simons 的论文<sup>[1]</sup>和陈先生的讲义<sup>[3]</sup>对于体积泛函的第一、第二变分计算过于技巧化,不容易推广到其它复杂泛函。为了解决这个问题,胡泽军、李海中、Reilly 等三位学者做出了基本贡献。国内郑州大学胡泽军教授与清华大学李海中教授大胆假设、小心求证,利用拉回运算以及流形上微分算子的关系推导出了余标架场和第二基本型分量的变分公式,这组变分公式在复杂泛函的变分计算之中具有基础性的作用,能够极大简化计算过程<sup>[4]</sup>。Reilly<sup>[5]</sup>设计了超曲面第二基本型曲率张量的牛顿变换,并计算了空间形式之中超曲面的极度抽象的泛函的第一变分,在高余维数情形,李海中教授利用 Reilly 的思想推广了牛顿变换,研究了它的各种代数性质和变分性质,并成功应用于高阶极小子流形的研究。目前,第二基本型曲率张量、黎曼流形曲率张量的牛顿变换在子流形变分理论、共形几何变分理论以及大范围微分几何之中发挥了越来越重要的作用。本书第四章利用 Reilly、李海中的思想进一步发展了子流形第二基本型曲率张量的牛

顿变换理论,在某种程度上可以认为推广到了最一般的情形。

微分几何之中有著名的 Willmore 泛函,其定义如下

$$W_{(n, \frac{n}{2})} = \int_M \rho^{\frac{n}{2}} dv,$$

其中  $\rho = S - nH^2$  是迹零第二基本型  $\hat{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij}$  的模长平方。微分几何学家对泛函  $W_{(n, \frac{n}{2})}$  的下界估计有系列推测,现在学术界冠以美国数学家 Willmore 的名称<sup>[6-9]</sup>。

Chen B Y 指出泛函  $W_{(n, \frac{n}{2})}$  是共形不变泛函<sup>[10]</sup>。北京大学(福建师范大学)王长平教授从 Moebius 几何的角度出发,定义了球面中子流形完全的共形不变量谱系,此谱系包含几何不变量  $\rho$ , 因此 Willmore 泛函是共形不变泛函就是 Moebius 几何的一个自然推论<sup>[11]</sup>。王长平与团队的李同柱、马翔、王鹏、聂圣智等成员按照 Moebius 几何的纲领对 Willmore 子流形特别是 Willmore 曲面进行了全方位的研究<sup>[11-23]</sup>。Pinkall 对 Willmore 类型的子流形推导了一些不等式,对讨论间隙现象有一定用处<sup>[24]</sup>。清华大学李海中教授按照欧式不变量体系分别计算了 Willmore 泛函的第一变分公式,推导了 Simons 类积分不等式,决定出了间隙端点对应的特殊子流形,构造了与 Clifford 环面对偶的 Willmore 环面<sup>[25-27]</sup>。郭正教授利用郑绍远-丘成桐在文章<sup>[28]</sup>发明的一种自伴算子研究了 Willmore 泛函,得到了和李海中相似的关于间隙现象的结果。在曲面情形,李海中教授和德国的 Udo Simon 教授讨论了多种曲面特别是 Willmore 曲面的量子化现象<sup>[29]</sup>。

在 Willmore 子流形的构造方面,李海中教授、胡泽军教授、Luc、Vranken 三人合作,分别利用复欧式空间 Lagrange 球面和两条曲线的张量乘法实现了不平凡例子<sup>[30,31]</sup>。唐梓洲教授与严文娇等<sup>[32,33]</sup>利用等参函数、Clifford 代数与代数拓扑构造了 Willmore 超曲面的例子,非常深刻。华南师范大学魏国新教授利用常微分方程研究旋转超曲面的方法已经成功应用于 Willmore 类型子流形的构造<sup>[34,35]</sup>。

为了研究经典 Willmore 子流形的稳定性,Palmer 对 Willmore 曲面的共形 Gauss 映射和稳定性之间的关系进行了研究,并计算了第二变分公式<sup>[36,37]</sup>,李海中教授、王长平教授以及云南师范大学的郭正教授对于高维情形的 Willmore 泛函  $W_{(n, \frac{n}{2})}$  计算了第二变分公式,在此基础上证明了 Willmore 环面的稳定性<sup>[38]</sup>。

Willmore 泛函如此重要,使得微分几何学家对它有很多的推广。Cai M 对曲面情形研究了所谓的  $p$ -Willmore 泛函  $W_{(2,p)} = \int_M \rho^p dv$ 。在一定条件下建立了一些重要的积分不等式<sup>[39]</sup>。

郭正教授、李海中教授、浙江大学许洪伟教授对 extreme-Willmore 泛函  $W_{(n,1)} = \int_M \rho dv$  进行了研究,计算了第一变分公式,建立了积分不等式,讨论了点

点间隙现象和整体间隙现象,刻画了 Clifford 环面和 Veronese 曲面<sup>[40]</sup>。

中国人民大学吴兰教授对幂函数形式的 Willmore 泛函  $W_{(n,r)} = \int_M \rho^r dv$  做了同样的研究流程,统一了上文所述的 Willmore 泛函  $W_{(n,\frac{n}{2})}$  和 extremal Willmore 泛函  $W_{(n,1)}$  以及曲面情形的  $p$ -Willmore 泛函  $W_{(2,p)}$ <sup>[41]</sup>。

受上面的启发,本书作者提出了  $F$ -Willmore 泛函的概念  $W_{(n,F)} = \int_M F(\rho) dv$  得到了抽象的第一变分公式、抽象的 Simons 类积分不等式、抽象的间隙现象,决定出了间隙端点的特殊子流形,在某种意义上统一了前面的所有结果<sup>[94]</sup>。

通过超曲面的共形变换,我们知道  $\dot{S}_r = \frac{1}{r!} \delta_{i_1 \dots i_r}^{j_1 \dots j_r} h_{i_1 j_1} \dots h_{i_r j_r}$ ,  $\dot{S}_2 = -\rho$  是高阶的共形不变量,Willmore 不变量  $\rho$  是特殊情形。郭正教授对此展开了系统的研究,构造了高阶共形不变泛函  $W_{(r;n,\frac{n}{r})} = \int_M |\dot{S}_r|^{\frac{n}{r}} dv$ 。计算了上面泛函的第一变分公式,利用两条曲线的张量乘法实现了泛函临界点的不平凡例子的构造<sup>[43]</sup>。

其它方面,周家足教授发表的系列论文基于凸几何理论,对凸闭的超曲面的 Willmore 泛函的下界进行了估计,特别是决定了等式成立时的特殊超曲面<sup>[44-47]</sup>。浙江大学许洪伟教授利用特殊的 Schoerdingier 算子的特征值刻画了 Willmore 子流形。李海中教授与吴兰教授在文中建立了 Willmore 泛函与子流形的 Weyl 泛函的关系,决定出了等号成立的特殊子流形,这是一项具有创造性的工作<sup>[48]</sup>。四川师范大学的马志圣先生探讨了各阶 Willmore 泛函与 Betti 数的关系<sup>[49-52]</sup>。李海中教授与魏国新教授实现了对 6 维带有近 kaehler 结构(Caley 数定义)的单位球面之中的 Lagrange-Willmore 子流形的分类<sup>[53]</sup>。罗勇对 5 维单位球面之中的 Legendrian 稳定曲面和 Legendrian-Willmore 曲面建立了 Simons 类积分不等式,讨论了间隙现象<sup>[54]</sup>。文章[55-65]对具体的空间如四维欧式空间,三维、四维单位球面,Whitney 球面,复射影空间,乘积空间,Lagrangian 环面之中的 Willmore 泛函或者 Willmore 子流形进行了具体的研究。文章[66-72]对 Willmore 子流形的几何性质特别是对称性或者不变量做了研究,如 Peter-Li 和丘成桐定义的共形不变量,Willmore 曲面的对偶性、可比较性,共形不变 Gauss 映射, Bernstein 性质等。

文章[73-79]回归到对泛函本身的几何测度论或者变分法的研究,特别是文章[79]宣称用 Min-Max 方法解决了 2 维情形的 Willmore 猜想。此外还有其他学者的工作,在此不一一列举。

Willmore 不变量  $\rho$  刻画了子流形与全脐子流形的局部差异,Willmore 泛函  $W_{(n,\frac{n}{2})} = \int_M \rho^{\frac{n}{2}} dv$  刻画了子流形与全脐子流形的整体差异。我们知道曲率模长  $S$  刻画了子流形与全测地子流形的差异,所以一个自然的问题是可否按照 Willmore 泛

函的研究流程与思路来研究曲率模长的抽象泛函  $\int_M F(S) dv$ 。

本书的目的在于系统运用变分法、采用与 Willmore 泛函类似的思路系统研究各类曲率模长泛函，特别是本书作者提出的  $\int_M F(S) dv$  泛函，研究其第一变分公式，根据第一变分公式综合运用代数、方程等手段构造临界子流形的例子，计算其第二变分公式，讨论临界子流形的稳定性，最重要的是利用推导出来的 Simons 类积分不等式研究临界子流形的间隙现象，决定出间隙端点对应的特殊子流形。本书的意义在于丰富了子流形的泛函，加深了对子流形间隙现象的理解，表明间隙现象不仅仅在极小子流形和 Willmore 子流形上面发生，而且是一大类泛函临界点的普遍现象。

## 第2章 黎曼几何基本理论

本章列出需要的预备知识。包括：微分流形的定义、黎曼度量的存在性、黎曼几何的基本方程、共形变换公式。

### 2.1 微分流形的定义

本节回顾微分流形与黎曼几何基本方程，可以参见陈省身的讲义<sup>[2]</sup>或者教材<sup>[101]</sup>。

流形的概念是欧式空间的推广。粗略地说，流形在其上每一点的附近与欧式空间的一个开集是同胚的，因此在每一点的附近可以引进局部坐标系。流形可以说是一块一块的“欧式空间”粘起来的结果。流形之内的坐标是局部的，本身没有多大的意义；流形研究的主要目的是经过坐标卡的变换而保持不变的性质。这是与欧氏空间中的数学对象不同的地方。

为了描述清楚流形的定义，我们需要拓扑空间和欧式空间的概念。

**定义 2.1:** 假设  $X$  是任意一个集合，我们用符号  $2^X = P(X)$  表示集合  $X$  的所有子集组成的集合， $A$  是空间  $2^X$  的一个子集，也就是  $X$  其上的一个子集族。

$$A \subset 2^X.$$

如果子集族  $A$  满足如下性质，我们称  $A$  为  $X$  上的拓扑结构。

1.  $\emptyset, X \in A$ ;
2.  $\forall O_i \in A, i \in I, \bigcup_{i \in I} O_i \in A$ ;
3.  $\forall O_i \in A, i = 1, 2, \dots, n, \bigcap_{i=1}^n O_i \in A$ .

集合族  $A$  之中的元素称为开集。对于一个点  $x \in X$  和一个开集  $O$ ，如果  $x \in O$ ，我们称开集  $O$  为点  $x$  的领域。

**定义 2.2:** 假设  $(X, A)$  是任意一个拓扑空间，任取集合  $X$  之中的两个元素  $x, y$ ，如果存在开集  $O_1, O_2$ ，满足如下条件，我们称空间  $(X, A)$  是 Hausdorff 的。

$$x \in O_1, y \in O_2, O_1 \cap O_2 = \emptyset.$$

我们用  $\mathbb{R}$  表示实数域。用  $\mathbb{R}^m$  表示  $m$  维的实数空间。

$$\mathbb{R}^m = \{x = (x_1, x_2, \dots, x_m) | x_i \in \mathbb{R}, 1 \leq i \leq m\},$$

即  $\mathbb{R}^m$  是全体有序的  $m$  个实数所形成的数组的集合，实数  $x_i$  表示点  $x \in \mathbb{R}^m$  的第  $i$



个坐标。对于任意的  $x, y \in \mathbb{R}^m$ ,  $a \in \mathbb{R}$ , 我们定义

$$(x+y)_i = x_i + y_i;$$

$$(ax)_i = ax_i.$$

于是在  $\mathbb{R}^m$  上, 我们定义了线性结构, 从而  $\mathbb{R}^m$  成为  $m$  维向量空间。

$\mathbb{R}^m$  上除了有线性结构以外, 还有距离结构或者说拓扑结构。对于  $\mathbb{R}^m$  之中的两个点  $x, y \in \mathbb{R}^m$ , 我们定义

$$|x| = \sqrt{\sum_{i=1}^n (x_i)^2}, |x-y| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2};$$

$$d(x, y) = |x - y| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

可以验证, 函数  $d(x, y)$  满足距离定义的三个公理。

$$d(x, y) \geq 0, d(x, y) = 0 \text{ 当且仅当 } x = y;$$

$$d(x, y) = d(y, x);$$

$$d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in \mathbb{R}^m.$$

所以, 函数  $d(x, y)$  是  $\mathbb{R}^m$  之中的距离函数。我们可以定义距离空间  $\mathbb{R}^m$  的拓扑基

$$B(x, r) = \{y : d(y, x) < r\}, \forall x \in \mathbb{R}^m, r > 0.$$

空间  $\mathbb{R}^m$  之中的开集为任意多个开球的并集。

于是, 以上定义了线性结构与距离结构的空间  $\mathbb{R}^m$  被称为欧式空间。

**定义 2.3:** 假设  $M$  是 Hausdorff 空间。若对任意一点  $x \in M$ , 都有  $x$  在  $M$  之中的一个领域  $U$  同胚于  $m$  维欧式空间  $\mathbb{R}^m$  的一个开集, 则称  $M$  是一个  $m$  维拓扑流形。

我们假设上面定义之中提到的同胚映射是

$$\phi_U : U \rightarrow \phi_U(U),$$

这里

$$\phi_U(U)$$

是欧式空间之中的开集, 则称  $(U, \phi_U(U))$  是拓扑流形  $M$  的一个坐标卡。因为  $\phi_U$  是同胚, 所以对任意一点  $y \in U$ , 可以把  $\phi_U(y) \in \mathbb{R}^m$  的坐标定义为  $y$  的坐标, 即命

$$u_i = (\phi_U(y))_i, y \in U, i = 1, \dots, m,$$

我们称  $u_i (i = 1, 2, \dots, m)$  为点  $y \in U$  的局部坐标。

设  $(U_\alpha, \phi_\alpha)$  和  $(U_\beta, \phi_\beta)$  是流形之中的两个坐标卡, 设  $V_\alpha = \phi_\alpha(U_\alpha)$ ,  $V_\beta = \phi_\beta(U_\beta)$  为欧式空间之中对应的开集。那么两个坐标卡之间的关系可能出现有

- $U_\alpha \cap U_\beta = \emptyset$ , 此时我们称坐标卡  $(U_\alpha, \phi_\alpha)$  和  $(U_\beta, \phi_\beta)$  是任意相容的。
- $U_\alpha \cap U_\beta \neq \emptyset$ , 此时  $V_{\alpha:\beta} = \phi_\alpha(U_\alpha \cap U_\beta)$  和  $V_{\beta:\alpha} = \phi_\beta(U_\beta \cap U_\alpha)$  是欧式空间之中的非空开集; 显然下面两个映射是同胚:

$$\phi_\beta \cdot \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\beta \cap U_\alpha);$$

$$\phi_\alpha \cdot \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\beta \cap U_\alpha).$$

如果上面的映射都是  $C^r$  ( $r$  次连续可微) 或者  $C^\infty$  (光滑) 或者  $C^\omega$  (解析) 的, 我们称坐标卡  $(U_\alpha, \phi_\alpha)$  和  $(U_\beta, \phi_\beta)$  是  $C^r$  或者  $C^\infty$  或者  $C^\omega$  相容的。

**定义 2.4:** 假设  $M$  是一个  $m$  维的拓扑流形。如果在  $M$  上给定了一个坐标卡集合  $C = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ , 满足如下条件, 则称  $C$  为  $M$  上的一个  $C^r$  ( $C^\infty, C^\omega$ ) 微分机构:

1.  $\{U_\alpha\}_{\alpha \in A}$  是流形  $M$  的一个开覆盖;

2.  $C$  之中的任意两个坐标卡都是  $C^r$  ( $C^\infty, C^\omega$ ) 相容的;

3.  $C$  是极大的, 即:  $M$  的任意一个坐标卡  $(U, \phi_U)$ , 如果与  $C$  之中的任意一个坐标卡都是  $C^r$  ( $C^\infty, C^\omega$ ) 相容的, 那么此坐标卡  $(U, \phi_U) \in C$ 。

在光滑流形之上, 光滑函数的定义是有意义的。设函数  $f$  是定义在  $m$  维光滑流形  $M$  之上的实函数。若点  $x \in M$ ,  $(U_\alpha, \phi_\alpha)$  是包含点  $x$  的容许坐标卡, 那么函数

$$f \cdot \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \rightarrow \mathbb{R}$$

是定义在欧式空间  $\mathbb{R}^m$  的开集  $\phi_\alpha(U_\alpha)$  之上的实函数。如果函数  $f \cdot \phi_\alpha^{-1}$  是光滑的, 我们称函数  $f$  在点  $x$  是光滑的, 如果函数在流形  $M$  上的每一点都是光滑的, 则称函数  $f$  在整个流形之上都是光滑的。

函数  $f$  在一点  $x$  的光滑性实际上与点  $x$  的容许坐标卡的选择无关。假设点  $x$  有两个容许坐标卡

$$(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta),$$

那么我们有

$$U_\alpha \cap U_\beta \neq \emptyset, \phi_\alpha \cdot \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \in C^\infty.$$

于是

$$f \cdot \phi_\beta^{-1} = f \cdot \phi_\alpha^{-1} \cdot \phi_\alpha \cdot \phi_\beta^{-1}$$

在点  $x$  也是光滑的, 因此光滑性的定义与局部容许坐标卡的选择无关。

对于光滑流形  $M$  上的每一点  $x$ , 都有很多通过它的光滑曲线, 通过容许坐标卡, 我们可以知道, 通过点  $x$  的光滑曲线在局部上就是空间  $\mathbb{R}^m$  的曲线, 于是可以微分计算曲线的切线, 并且满足坐标卡的变化规律。此类切线集合起来, 可以认为是点  $x$  的切空间, 记为  $T_x M$ 。所有的切空间集合起来, 可以认为是切丛, 记为  $TM$ 。

在每一点  $x$  的切空间  $T_x M$ , 我们定义其上的正定对称二次型, 记为  $G$ , 在局部标架之下, 可以表示为

$$ds^2(x) = G_x = g_{ij}(x) du^i \otimes du^j.$$

如果可以整体光滑地定义于整个流形之上, 那么我们说  $G$  是流形  $M$  的黎曼度量。

通过单位分解函数, 我们可以构造出整个流形之上的黎曼度量。实际上, 我们有下面的著名的命题。

**命题 2.1:** 任意的  $m$  维光滑流形之上必有黎曼度量。

## 2.2 黎曼几何结构方程

有了流形之上的黎曼度量, 我们就可以推导黎曼几何的基本方程。

设  $(N, ds^2)$  是黎曼流形,  $S = (s_1, \dots, s_N)^t$  和  $\sigma = (\sigma^1, \dots, \sigma^N)$  分别是  $TN$ ,  $T^*N$  的局部正交标架, 显然有

$$S \cdot S^t = I, \quad \sigma^i \cdot \sigma = I.$$

设  $D$  是联络,  $\omega, \tau, \Omega$  分别是联络形式、挠率形式和曲率形式, 那么有以下方程。

- 运动方程

$$DS = \omega \otimes S, \quad DS_A = \omega_A^B \otimes S_B = \Gamma_{AC}^B \sigma^C \otimes S_B, \quad (2-1)$$

$$D\sigma = -\sigma \otimes \omega, \quad D\sigma^A = -\sigma^B \otimes \omega_B^A. \quad (2-2)$$

- 挠率方程

$$D(\sigma \otimes S) = d\sigma \otimes S - \sigma \wedge \omega \otimes S = (d\sigma - \sigma \wedge \omega) \otimes S = \tau \otimes S, \quad (2-3)$$

$$\tau = d\sigma - \sigma \wedge \omega, \quad \tau^A = d\sigma^A - \sigma^B \wedge \omega_B^A. \quad (2-4)$$

- 曲率方程

$$D^2 S = D(\omega \otimes S) = (d\omega - \omega \wedge \omega) \otimes S = \Omega \otimes S, \quad (2-5)$$

$$D^2 S_A = \frac{1}{2} R_{ACD}^B \sigma^C \wedge \sigma^D \otimes S_B, \quad (2-6)$$

$$D^2 \sigma = D(-\sigma \otimes \omega) = -\sigma \otimes (d\omega - \omega \otimes \omega) = -\sigma \otimes \Omega. \quad (2-7)$$

- 第一 Bianchi 方程

$$D^2(\sigma \otimes S) = \sigma \wedge D^2 S = (\sigma \wedge \Omega) \otimes S, \quad (2-8)$$

$$D^2(\sigma \otimes S) = D(D(\sigma \otimes S)) = D(\tau \otimes S) = (d\tau + \tau \wedge \omega) \otimes S, \quad (2-9)$$

$$d\tau + \tau \wedge \omega = \sigma \wedge \Omega. \quad (2-10)$$

- 第二 Bianchi 方程

$$D^3 S = D(D^2(S)) = D(\Omega \otimes S) = (d\Omega + \Omega \wedge \omega) \otimes S, \quad (2-11)$$

$$D^3 S = D^2(\omega \otimes S) = \omega \wedge \Omega \otimes S, \quad (2-12)$$

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega. \quad (2-13)$$

- 相容方程

$$0 = DI = D(S \cdot S^t) = \omega \otimes S \cdot S^t + S \cdot S^t \otimes \omega^t = \omega + \omega^t, \quad (2-14)$$

$$0 = D^2 I = D^2(S \cdot S^t) = \Omega \otimes S \cdot S^t + S \cdot S^t \otimes \Omega^t = \Omega + \Omega^t. \quad (2-15)$$

**注释 2.1:** 在本书中, 作者采用活动标架法, 故约定:  $S_A = S^A$ ,  $\sigma^A = \sigma_A$ ,  $\tau^A = \tau_A$ ,  $\omega_A^B = \omega_{AB}$ ,  $\Gamma_{AC}^B = \Gamma_{ABC}$ ,  $\Omega_A^B = \Omega_{AB}$ ,  $R_{ACD}^B = R_{ABCD}$ .

黎曼联络由相容方程和挠率为零条件唯一决定, 这就是著名的黎曼联络存在唯一定理。

**定理 2.1:** 设  $(N, ds^2)$  是黎曼流形,  $\sigma$  是局部正交余标架, 那么黎曼联络  $\omega$  由以下方程唯一决定。

$$\omega + \omega^t = 0, \quad d\sigma - \sigma \wedge \omega = 0.$$

对于黎曼流形上的任意张量  $T$ ,

$$T = T_{j_1 \cdots j_s}^{i_1 \cdots i_r} \sigma^{j_1} \otimes \cdots \otimes \sigma^{j_s} \otimes S_{i_1} \otimes \cdots \otimes S_{i_r}.$$

定义其协变导数如下:

$$DT_{j_1 \cdots j_s}^{i_1 \cdots i_r} = \sum_k T_{j_1 \cdots j_s}^{i_1 \cdots i_r, k} \sigma^k \quad (2-16)$$

$$= dT_{j_1 \cdots j_s}^{i_1 \cdots i_r} - \sum_{1 \leq a \leq s} T_{j_1 \cdots p \cdots j_s}^{i_1 \cdots i_r} \omega_{ja}^p + \sum_{1 \leq b \leq r} T_{j_1 \cdots j_s}^{i_1 \cdots p \cdots i_r} \omega_p^{i_b},$$

$$DT_{j_1 \cdots j_s, k}^{i_1 \cdots i_r} = \sum_l T_{j_1 \cdots j_s, k}^{i_1 \cdots i_r, l} \sigma^l$$

$$= dT_{j_1 \cdots j_s, k}^{i_1 \cdots i_r} - \sum_{1 \leq a \leq s} T_{j_1 \cdots p \cdots j_s, k}^{i_1 \cdots i_r} \omega_{ja}^p - T_{j_1 \cdots j_s, p}^{i_1 \cdots i_r} \omega_k^p + \sum_{1 \leq b \leq r} T_{j_1 \cdots j_s, k}^{i_1 \cdots p \cdots i_r} \omega_p^{i_b}. \quad (2-17)$$

有 Ricci 恒等式

$$T_{j_1 \cdots j_s, kl}^{i_1 \cdots i_r} - T_{j_1 \cdots j_s, lk}^{i_1 \cdots i_r} = \sum_a T_{j_1 \cdots p \cdots j_s}^{i_1 \cdots i_r} R_{ja kl}^p - \sum_b T_{j_1 \cdots j_s, k}^{i_1 \cdots p \cdots i_r} R_{pkl}^{i_b}. \quad (2-18)$$

特别的, 从曲率张量出发, 可以定义新的张量和函数

$$Ric = R_{ij} \sigma^i \otimes \sigma^j = \left( \sum_p R_{ipj}^p \right) \sigma^i \otimes \sigma^j, \quad (2-19)$$

$$R = \sum_i R_{ii} = \sum_{ij} R_{ijji}. \quad (2-20)$$

从 Bianchi 方程、相容方程出发可以得到下面的定理。

**定理 2.2:** 设  $(N, ds^2)$  是黎曼流形, 其上的曲率张量、Ricci 张量、数量曲率满足

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klji}, \quad (2-21)$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \quad (2-22)$$

$$R_{ijk l, h} + R_{ijlh, k} + R_{ijhk, l} = 0, \quad (2-23)$$

$$\sum_j R_{ij, j} = \frac{1}{2} R_{, i}. \quad (2-24)$$

特别地, 完备、单连通、常截面曲率  $c$  的空间记为  $R^n(c)$ , 满足以下简单关系:

$$R_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad (2-25)$$

$$R_{AB} = (n-1)c\delta_{AB}, \quad R = n(n-1)c. \quad (2-26)$$

## 2.3 共形几何变换公式

本节中主要回顾共形几何的基本知识, 参见文献[102]。为了方便起见, 引进一个符号称为 Kulkarni-Nomizu 乘法: 设  $a, b$  是两个对称的  $(0, 2)$  型张量, 通过以下运算构造  $(0, 4)$  型张量  $a \otimes b$ :

$$(a \otimes b)_{ijkl} = (a_{ik}b_{jl} - a_{il}b_{jk} + b_{ik}a_{jl} - b_{il}a_{jk}).$$

设  $(N, ds^2)$  是黎曼流形,  $S = (S_1, \dots, S_N)'$ ,  $\sigma = (\sigma^1, \dots, \sigma^N)$  是局部正交标架, 那么  $ds^2 = \sum_A (\sigma^A)^2$ 。设  $u$  是  $N$  上的光滑函数, 考虑  $N$  上新的度量  $\widetilde{ds}^2 = e^{2u} ds^2$ 。本节的目的就是要推导  $(N, \widetilde{ds}^2)$ ,  $(N, ds^2)$  之间的各种张量之间的关系。约定:  $(N, \widetilde{ds}^2)$  上的各种量都加上  $\tilde{R}$ 。显然,  $\tilde{S} = \frac{S}{e^u}$ ,  $\tilde{\sigma} = e^u \sigma$  是  $(N, \widetilde{ds}^2)$  的局部正交标架。

设  $\tilde{\omega}$  是  $(N, \widetilde{ds}^2)$  的联络形式, 设  $\omega$  是  $(N, ds^2)$  的联络形式。根据定理 2.1, 有

$$du = \sum_A u_{,A} \sigma^A, \quad (2-27)$$

$$\omega + \omega^i = 0, \quad d\sigma - \sigma \wedge \omega = 0, \quad (2-28)$$

$$\tilde{\omega} + \tilde{\omega}^i = 0, \quad d\tilde{\sigma} - \tilde{\sigma} \wedge \tilde{\omega} = 0. \quad (2-29)$$

事实上, 有

$$d\tilde{\sigma}^A = d(e^u \sigma^A) = e^u \sum_B u_{,B} \sigma^B \wedge \sigma^A + e^u d\sigma^A \quad (2-30)$$

$$= e^u \sum_B u_{,B} \sigma^B \wedge \sigma^A + e^u \sigma^B \wedge \omega_B^A \quad (2-31)$$

$$= \sum_B \tilde{\sigma}^B \wedge (u_{,B} \sigma^A + \omega_B^A) \quad (2-32)$$

$$= \sum_B \tilde{\sigma}^B \wedge (u_{,B} \sigma^A - u_{,A} \sigma^B + \omega_B^A). \quad (2-33)$$

显然, 我们证明了下面的定理。

**定理 2.3:** 在  $(N, \widetilde{ds}^2)$  和  $(N, ds^2)$  的标架  $\tilde{S}, S$  下, 有如下关系

$$\tilde{\omega}_A^B = u_{,A} \sigma^B - u_{,B} \sigma^A + \omega_A^B, \quad (2-34)$$

$$\tilde{\Gamma}_{AC}^B = \frac{u_{,A}}{e^u} \delta_{BC} - \frac{u_{,B}}{e^u} \delta_{AC} + \frac{\Gamma_{AC}^B}{e^u}. \quad (2-35)$$

设  $\tilde{\Omega}$  是  $(N, \widetilde{ds}^2)$  的曲率形式, 而  $\Omega$  是  $(N, ds^2)$  的曲率形式。

$$du = \sum_A u_{,A} \sigma^A, Du_{,A} = du_{,A} - u_{,C} \omega_A^C = u_{,AB} \sigma^B, \quad (2-36)$$

$$\Omega + \Omega' = 0, d\omega - \omega \wedge \omega = \Omega, \quad (2-37)$$

$$\tilde{\Omega} + \tilde{\Omega}' = 0, d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega} = \tilde{\Omega}. \quad (2-38)$$

由定理 2.3, 有

$$\frac{1}{2} \tilde{R}_{ABCD} \tilde{\sigma}^C \wedge \tilde{\sigma}^D = \tilde{\Omega}_A^B = d\tilde{\omega}_A^B - \tilde{\omega}_A^C \wedge \tilde{\omega}_C^B \quad (2-39)$$

$$= d(u_{,A} \sigma^B - u_{,B} \sigma^A + \omega_A^B) - (u_{,A} \sigma^C - u_{,C} \sigma^A + \omega_A^C) \wedge (u_{,B} \sigma^B - u_{,B} \sigma^C + \omega_C^B) \quad (2-40)$$

$$= du_{,A} \sigma^B + u_{,A} d\sigma^B - du_{,B} \sigma^A - u_{,B} d\sigma^A + d\omega_A^B - u_{,A} u_{,C} \sigma^C \wedge \sigma^B - u_{,A} \sigma^C \wedge \omega_C^B + u_{,B} u_{,C} \sigma^A \wedge \sigma^B - u_{,C} u_{,B} \sigma^A \wedge \sigma^C + u_{,C} \sigma^A \wedge \omega_C^B - u_{,C} \omega_A^C \wedge \sigma^B + u_{,B} \omega_A^C \wedge \sigma^C - \omega_A^C \wedge \omega_C^B \quad (2-41)$$

$$= (du_{,A} - u_{,C} \omega_A^C) \wedge \sigma^B + u_{,A} (d\sigma^B - \sigma^C \wedge \omega_C^B) - (du_{,B} - u_{,C} \omega_B^C) \wedge \sigma^A - u_{,B} (d\sigma^A - \sigma^C \wedge \omega_C^A) - u_{,A} u_{,C} \sigma^C \wedge \sigma^B + |Du|^2 \sigma^A \wedge \sigma^B - u_{,C} u_{,B} \sigma^A \wedge \sigma^C + d\omega_A^B - \omega_A^C \wedge \omega_C^B \quad (2-42)$$

$$= u_{,AC} \sigma^C \wedge \sigma^B - u_{,BC} \sigma^C \wedge \sigma^A - u_{,A} u_{,C} \sigma^C \wedge \sigma^B + |Du|^2 \sigma^A \wedge \sigma^B - u_{,C} u_{,B} \sigma^A \wedge \sigma^C + \Omega_A^B \quad (2-43)$$

$$= u_{,AC} \delta_{BD} \sigma^C \wedge \sigma^D - u_{,BC} \delta_{AD} \sigma^C \wedge \sigma^D - u_{,A} u_{,C} \delta_{BD} \sigma^C \wedge \sigma^D + u_{,B} u_{,C} \delta_{AD} \sigma^C \wedge \sigma^D + |Du|^2 \delta_{AC} \delta_{BD} \sigma^C \wedge \sigma^D + \frac{1}{2} R_{ABCD} \sigma^C \wedge \sigma^D \quad (2-44)$$

$$= \frac{1}{2} \frac{1}{e^{2u}} (u_{,AC} \delta_{BD} - u_{,BC} \delta_{AD} + \delta_{AC} u_{BD} - \delta_{BC} u_{,AD}) \tilde{\sigma}^C \wedge \tilde{\sigma}^D - \frac{1}{2} \frac{1}{e^{2u}} (u_{,A} u_{,C} \delta_{BD} - u_{,B} u_{,C} \delta_{AD} + \delta_{AC} u_{,B} u_{,D} - \delta_{BC} u_{,A} u_{,D}) \tilde{\sigma}^C \wedge \tilde{\sigma}^D + \frac{1}{2} \frac{1}{e^{2u}} |Du|^2 (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}) \tilde{\sigma}^C \wedge \tilde{\sigma}^D + \frac{1}{2} \frac{1}{e^{2u}} R_{ABCD} \tilde{\sigma}^C \wedge \tilde{\sigma}^D \quad (2-45)$$

$$= \frac{1}{2} \left[ \frac{1}{e^{2u}} (D^2 u \otimes \delta)_{ABCD} \tilde{\sigma}^C \wedge \tilde{\sigma}^D \right] - \frac{1}{2} \left[ \frac{1}{e^{2u}} ((Du \otimes Du) \otimes \delta)_{ABCD} \tilde{\sigma}^C \wedge \tilde{\sigma}^D \right] + \frac{1}{2} \left[ \frac{|Du|^2}{e^{2u}} (\delta \otimes \delta)_{ABCD} \tilde{\sigma}^C \wedge \tilde{\sigma}^D \right] + \frac{1}{2} \left[ \frac{1}{e^{2u}} R_{ABCD} \tilde{\sigma}^C \wedge \tilde{\sigma}^D \right]. \quad (2-46)$$

这样, 用活动标架法证明了下面的定理.

**定理 2.4:** 在  $(N, \widetilde{ds}^2)$  和  $(N, ds^2)$  的标架  $\tilde{S}$ ,  $S$  下, 曲率张量或函数的变化规律为

$$\widetilde{Riem} = \frac{1}{e^{2u}} \left[ (D^2 u - Du \otimes Du) \otimes \delta + \frac{|Du|^2}{2} \delta \otimes \delta + Riem \right], \quad (2-47)$$

$$\tilde{R}_{ABCD} = \frac{1}{e^{2u}} \{ [u_{,AC} \delta_{BD} - u_{,BC} \delta_{AD} + \delta_{AC} u_{BD} - \delta_{BC} u_{,AD}]$$

$$- [u_{,A}u_{,C}\delta_{BD} - u_{,B}u_{,C}\delta_{AD} + \delta_{AC}u_{,B}u_{,D} - \delta_{BC}u_{,A}u_{,D}] \\ + |Du|^2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + R_{ABCD} \}, \quad (2-48)$$

$$\widetilde{Ric} = \frac{1}{e^{2u}} [Ric - (n-2)(D^2u - Du \otimes Du) - (\Delta u + (n-2)|Du|^2)\delta], \quad (2-49)$$

$$\tilde{R}_{AB} = \frac{1}{e^{2u}} [R_{AB} - (n-2)u_{,AB} + (n-2)u_{,A}u_{,B} \\ - \Delta u \delta_{AB} - (n-2)|Du|^2\delta_{AB}], \quad (2-50)$$

$$\tilde{R} = \frac{1}{e^{2u}} [R - 2(n-1)\Delta u - (n-1)(n-2)|Du|^2]. \quad (2-51)$$

**注释 2.2:** 在上面的定理及其推导过程中, 对  $u$  的协变导数都是在  $(N, ds^2)$  的意义下进行的。

在黎曼几何中, 对曲率张量有以下著名的正交分解, 其中当  $n \geq 4$  时, 称  $W$  为 Weyl 张量。

$$Riem = U \oplus V \oplus W.$$

- 若  $\dim_R N = 2$ ,

$$U = -\frac{R}{4}(\delta \otimes \delta), \quad V = 0, \quad W = 0.$$

- 若  $\dim_R N = 3$ ,

$$U = -\frac{R}{12}(\delta \otimes \delta), \quad V = -(\text{Ric} - \frac{R}{3}\delta) \otimes \delta, \quad W = 0.$$

- 若  $\dim_R N \geq 4$ ,

$$U = -\frac{R}{2n(n-1)}\delta \otimes \delta, \quad V = -\frac{1}{n-2}(\text{Ric} - \frac{R}{n}\delta) \otimes \delta, \quad W = Riem - U - V.$$

记  $X = U \oplus V$ , 则

$$Riem = X \oplus W.$$

- 若  $\dim_R N = 2$ ,

$$X = -\frac{R}{4}(\delta \otimes \delta), \quad W = 0.$$

- 若  $\dim_R N = 3$ ,

$$X = -(\text{Ric} - \frac{R}{4}\delta) \otimes \delta, \quad W = 0.$$

- 若  $\dim_R N \geq 4$ ,

$$X = -\frac{1}{(n-2)}(\text{Ric} - \frac{R}{2(n-1)}\delta) \otimes \delta, \quad W = Riem - X.$$

当  $n \geq 3$  时, 定义 Schouten 张量和 Bak 张量分别为

$$Scht = (C_{ij}) =: \text{Ric} - \frac{R}{2(n-1)}\delta, \quad (2-52)$$

$$Bak = (B_{ijk}) = (R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)}(\delta_{ij}R_{,k} - \delta_{ik}R_{,j}) \quad (2-53)$$

$$= C_{ij,k} - C_{ik,j}. \quad (2-54)$$

显然, 当  $n \geq 3$  时,  $X = -\frac{1}{n-2}C \otimes \delta$ .

在  $(N, \widetilde{ds^2})$  和  $(N, ds^2)$  的标架  $\tilde{S}, S$  下, 直接计算有

$$\widetilde{Riem} = \tilde{U} \oplus \tilde{V} \oplus \tilde{W} = \tilde{X} \oplus \tilde{W}.$$

•  $\dim_R N = 2$

$$\tilde{U} = -\frac{\tilde{R}}{4}(\delta \otimes \delta) \quad (2-55)$$

$$= \frac{1}{e^{2u}}(-\frac{1}{4})(R - 2\Delta u)(\delta \otimes \delta) \quad (2-56)$$

$$= \frac{1}{e^{2u}}(-\frac{R}{4} + \frac{1}{2}\Delta u)(\delta \otimes \delta) \quad (2-57)$$

$$= \frac{1}{e^{2u}}[U + \frac{1}{2}\Delta u \delta \otimes \delta], \quad (2-58)$$

$$\tilde{V} = 0, \quad \tilde{W} = 0, \quad \tilde{X} = \frac{1}{e^{2u}}[X + \frac{1}{2}\Delta u \delta \otimes \delta]. \quad (2-59)$$

•  $\dim_R N = 3$

$$\tilde{U} = -\frac{\tilde{R}}{12}(\delta \otimes \delta) \quad (2-60)$$

$$= \frac{1}{e^{2u}}(-\frac{1}{12})(R - 4\Delta u - 2|Du|^2)(\delta \otimes \delta) \quad (2-61)$$

$$= \frac{1}{e^{2u}}(-\frac{R}{12} + \frac{1}{3}\Delta u + \frac{1}{6}|Du|^2)(\delta \otimes \delta) \quad (2-62)$$

$$= \frac{1}{e^{2u}}[U + \frac{1}{3}\Delta u \delta \otimes \delta + \frac{1}{6}|Du|^2 \delta \otimes \delta], \quad (2-63)$$

$$\tilde{V} = -(\tilde{Ric} - \frac{\tilde{R}}{3}\delta) \otimes \delta \quad (2-64)$$

$$\begin{aligned} &= \frac{-1}{e^{2u}}(Ric - |Du|^2 \delta - \Delta u \delta + Du \otimes Du - D^2 u - \frac{R}{3}\delta \\ &\quad + \frac{4}{3}\Delta u \delta + \frac{2}{3}|Du|^2 \delta) \otimes \delta \end{aligned} \quad (2-65)$$

$$= \frac{-1}{e^{2u}}(Ric - \frac{R}{3}\delta - \frac{1}{3}(|Du|^2 - \Delta u)\delta + Du \otimes Du - D^2 u) \otimes \delta \quad (2-66)$$



$$= \frac{1}{e^{2u}} \left[ V + \frac{1}{3} (|Du|^2 - \Delta u) \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta \right], \quad (2-67)$$

$$\widetilde{W} = 0, \quad \widetilde{X} = \frac{1}{e^{2u}} \left[ X + \frac{1}{2} |Du|^2 \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta \right]. \quad (2-68)$$

•  $\dim_R N \geq 4$

$$\widetilde{U} = -\frac{2n(n-1)}{e^{2u}} (\delta \otimes \delta) \quad (2-69)$$

$$= \frac{1}{e^{2u}} \left( -\frac{1}{2n(n-1)} \right) \quad (2-70)$$

$$\times (R - 2(n-1)\Delta u - (n-1)(n-2)|Du|^2) (\delta \otimes \delta) \quad (2-71)$$

$$= \frac{1}{e^{2u}} \left( -\frac{R}{2n(n-1)} + \frac{1}{n} \Delta u + \frac{n-2}{2n} |Du|^2 \right) (\delta \otimes \delta) \quad (2-72)$$

$$= \frac{1}{e^{2u}} \left[ U + \frac{1}{n} \Delta u \delta \otimes \delta + \frac{n-2}{2n} |Du|^2 \delta \otimes \delta \right], \quad (2-73)$$

$$\widetilde{V} = -\frac{1}{n-2} (\widetilde{Ric} - \frac{\widetilde{R}}{n} \delta) \otimes \delta \quad (2-74)$$

$$\begin{aligned} &= \frac{1}{e^{2u}} \left( -\frac{1}{n-2} \right) \times [Ric - (n-2)|Du|^2 \delta - \Delta u \delta \\ &\quad + (n-2)Du \otimes Du - (n-2)D^2 u - \frac{R}{n} \delta + \frac{2(n-1)}{n} \Delta u \delta \\ &\quad + \frac{(n-1)(n-2)}{n} |Du|^2 \delta] \otimes \delta \end{aligned} \quad (2-75)$$

$$\begin{aligned} &= \frac{1}{e^{2u}} \left[ -\frac{1}{n-2} \left( Ric - \frac{R}{n} \delta \right) + \frac{1}{n} (|Du|^2 - \Delta u) \delta \right. \\ &\quad \left. + D^2 u - Du \otimes Du \right] \otimes \delta \end{aligned} \quad (2-76)$$

$$= \frac{1}{e^{2u}} \left[ V + \frac{1}{n} (|Du|^2 - \Delta u) \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta \right], \quad (2-77)$$

$$\widetilde{X} = \frac{1}{e^{2u}} \left[ X + \frac{1}{2} |Du|^2 \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta \right], \quad (2-78)$$

$$\widetilde{Riem} = \frac{1}{e^{2u}} \left[ Riem + \frac{1}{2} |Du|^2 \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta \right], \quad (2-79)$$

$$\widetilde{W} = \widetilde{Riem} - \widetilde{X} = \frac{1}{e^{2u}} W. \quad (2-80)$$

•  $\dim_R N \geq 3$

$$\widetilde{Sch} = \widetilde{Ric} - \frac{\widetilde{R}}{2(n-1)} \delta \quad (2-81)$$

$$= \frac{1}{e^{2u}} \left[ Ric - \frac{R}{2(n-1)} \delta - \frac{n-2}{2} |Du|^2 \delta \right]$$

$$+ (n-2)(Du \otimes Du - D^2u)] \quad (2-82)$$

$$= \frac{1}{e^{2u}} [Schu - (n-2)(D^2u - Du \otimes Du) - \frac{n-2}{2}|Du|^2\delta]. \quad (2-83)$$

设  $\tilde{D}$ ,  $D$  分别是  $(N, ds^2)$ ,  $(N, \tilde{ds}^2)$  上的黎曼联络。定义曲率张量的协变导数如下:

• Ricci 曲率

$$D\tilde{R}_{AB} = \tilde{R}_{AB,C}\sigma^C = d\tilde{R}_{AB} - \tilde{R}_{PB}\omega_A^P - \tilde{R}_{AP}\omega_B^P, \quad (2-84)$$

$$\tilde{D}\tilde{R}_{AB} = \tilde{R}_{AB,\tilde{C}}\tilde{\sigma}^{\tilde{C}} = d\tilde{R}_{AB} - \tilde{R}_{PB}\tilde{\omega}_A^P - \tilde{R}_{AP}\tilde{\omega}_B^P \quad (2-85)$$

$$= d\tilde{R}_{AB} - \tilde{R}_{PB}(\omega_A^P + u_{,A}\sigma^P - u_{,P}\sigma^A) - \tilde{R}_{AP}(\omega_B^P + u_{,B}\sigma^P - u_{,P}\sigma^B) \quad (2-86)$$

$$= \tilde{R}_{AB,C}\sigma^C - \tilde{R}_{PB}u_{,A}\sigma^P + \tilde{R}_{PB}u_{,P}\sigma^A - \tilde{R}_{AP}u_{,B}\sigma^P + \tilde{R}_{AP}u_{,P}\sigma^B \quad (2-87)$$

$$= \tilde{R}_{AB,C}\sigma^C - (u_{,A}\tilde{R}_{BC} + u_{,B}\tilde{R}_{AC} - \delta_{AC}u_{,P}\tilde{R}_{PB} - \delta_{BC}u_{,P}\tilde{R}_{PA})\sigma^C \quad (2-88)$$

$$= \frac{1}{e^u} [\tilde{R}_{AB,C} - (u_{,A}\tilde{R}_{BC} + u_{,B}\tilde{R}_{AC} - \delta_{AC}u_{,P}\tilde{R}_{PB} - \delta_{BC}u_{,P}\tilde{R}_{PA})] \tilde{\sigma}^{\tilde{C}}. \quad (2-89)$$

• 数量曲率

$$\tilde{D}\tilde{R} = \tilde{R}_{,\tilde{\lambda}}\tilde{\sigma}^{\tilde{\lambda}} = d\tilde{R}, D\tilde{R} = \tilde{R}_{,A}\sigma^A = d\tilde{R}.$$

有

$$\tilde{R}_{AB,\tilde{C}} = \frac{1}{e^u} [\tilde{R}_{AB,C} - (u_{,A}\tilde{R}_{BC} + u_{,B}\tilde{R}_{AC} - \delta_{AC}u_{,P}\tilde{R}_{PB} - \delta_{BC}u_{,P}\tilde{R}_{PA})], \quad (2-90)$$

$$\tilde{R}_{,\tilde{\lambda}} = \frac{1}{e^u} \tilde{R}_{,A}. \quad (2-91)$$

对于 Bak 张量, 计算有

$$\widetilde{Bak} = \tilde{R}_{AB,\tilde{C}} - \tilde{R}_{AC,\tilde{B}} - \frac{1}{2(n-1)}(\delta_{AB}\tilde{R}_{,\tilde{C}} - \delta_{AC}\tilde{R}_{,\tilde{B}}) \quad (2-92)$$

$$\begin{aligned} &= \frac{1}{e^u} [\tilde{R}_{AB,C} - (u_{,A}\tilde{R}_{BC} + u_{,B}\tilde{R}_{AC} - \delta_{AC}u_{,P}\tilde{R}_{PB} - \delta_{BC}u_{,P}\tilde{R}_{PA})] \\ &\quad - \frac{1}{e^u} [\tilde{R}_{AC,B} - (u_{,A}\tilde{R}_{BC} + u_{,C}\tilde{R}_{AB} - \delta_{AB}u_{,P}\tilde{R}_{PB} - \delta_{BC}u_{,P}\tilde{R}_{PA})] \\ &\quad - \frac{1}{2(n-1)} \frac{1}{e^u} (\delta_{AB}\tilde{R}_{,C} - \delta_{AC}\tilde{R}_{,B}) \end{aligned} \quad (2-93)$$

$$\begin{aligned} &= \frac{1}{e^u} (\tilde{R}_{AB,C} - \tilde{R}_{AC,B} + u_{,C}\tilde{R}_{AB} - u_{,B}\tilde{R}_{AC} + \delta_{AC}u_{,P}\tilde{R}_{PB} - \delta_{AB}u_{,P}\tilde{R}_{PC}) \\ &\quad - \frac{1}{e^u} \frac{1}{2(n-1)} (\delta_{AB}\tilde{R}_{,C} - \delta_{AC}\tilde{R}_{,B}). \end{aligned} \quad (2-94)$$

对上式各项分别计算之

$$\begin{aligned}
u_{,C}\tilde{R}_{AB}-u_{,B}\tilde{R}_{AC} &= \frac{1}{e^{2u}}R_{AB}u_{,C}-R_{AC}u_{,B}+(n-2)u_{,AC}u_{,B}-(n-2)u_{,AB}u_{,C} \\
&\quad +\Delta u\delta_{AC}u_{,B}-\Delta u\delta_{AB}u_{,C} \\
&\quad + (n-2)|Du|^2\delta_{AC}u_{,B}-(n-2)|Du|^2\delta_{AB}u_{,C}\}. \quad (2-95)
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_{AB,C}-\tilde{R}_{AC,B} &= -2(u_{,C}\tilde{R}_{AB}-u_{,B}\tilde{R}_{AC})+\frac{1}{e^{2u}}\{R_{AB,C}-R_{AC,B}+(n-2)u_{,ACB}-(n-2)u_{,ABC} \\
&\quad +(n-2)u_{,AC}u_{,B}-(n-2)u_{,AB}u_{,C}+(\Delta u)_{,B}\delta_{AC}-(\Delta u)_{,C}\delta_{AB} \\
&\quad +2(n-2)u_{,p}u_{,pB}\delta_{AC}-2(n-2)u_{,p}u_{,pC}\delta_{AB}\}. \quad (2-96)
\end{aligned}$$

$$\begin{aligned}
\delta_{AC}u_{,p}\tilde{R}_{PB}-\delta_{AB}u_{,p}\tilde{R}_{PC} &= \frac{1}{e^{2u}}\{\delta_{AC}u_{,p}R_{PB}-\delta_{AB}u_{,p}R_{PC} \\
&\quad +(n-2)\delta_{AB}u_{,p}u_{,pC}-(n-2)\delta_{AC}u_{,p}u_{,pB} \\
&\quad +(n-2)\delta_{AC}|Du|^2u_{,B}-(n-2)\delta_{AB}|Du|^2u_{,C} \\
&\quad +\Delta u\delta_{AB}u_{,C}-\Delta u\delta_{AC}u_{,B} \\
&\quad +(n-2)|Du|^2\delta_{AB}u_{,C}-(n-2)|Du|^2\delta_{AC}u_{,B}\}. \quad (2-97)
\end{aligned}$$

$$\begin{aligned}
\delta_{AB}\tilde{R}_{,C}-\delta_{AC}\tilde{R}_{,B} &= \frac{1}{e^{2u}}\{[2\delta_{AC}u_{,B}R-2\delta_{AB}u_{,C}R+4(n-1)\delta_{AB}u_{,C}\Delta u-4(n-1)\delta_{AC}u_{,B}\Delta u \\
&\quad +2(n-1)(n-2)\delta_{AB}u_{,C}|Du|^2-2(n-1)(n-2)\delta_{AC}u_{,B}|Du|^2 \\
&\quad +[\delta_{AB}R_{,C}-\delta_{AC}R_{,B}+2(n-1)\delta_{AC}(\Delta u)_{,B}-2(n-1)\delta_{AB}(\Delta u)_{,C} \\
&\quad +2(n-1)(n-2)\delta_{AC}u_{,p}u_{,pB}-2(n-1)(n-2)\delta_{AB}u_{,p}u_{,pC}]\}. \quad (2-98)
\end{aligned}$$

综合以上的计算, 有

$$\begin{aligned}
\widehat{Bak} &= \frac{1}{e^{3u}}\{R_{AB,C}-R_{AC,B}+(n-2)u_{,ACB}-(n-2)u_{,ABC}+(n-2)u_{,AC}u_{,B} \\
&\quad -(n-2)u_{,AB}u_{,C}+(\Delta u)_{,B}\delta_{AC}-(\Delta u)_{,C}\delta_{AB}+2(n-2)u_{,p}u_{,pB}\delta_{AC} \\
&\quad -2(n-2)u_{,p}u_{,pC}\delta_{AB}-R_{AB}u_{,C}+R_{AC}u_{,B}-(n-2)u_{,AC}u_{,B}+(n-2)u_{,AB}u_{,C} \\
&\quad -\Delta u\delta_{AC}u_{,B}+\Delta u\delta_{AB}u_{,C}-(n-2)|Du|^2\delta_{AC}u_{,B}+(n-2)|Du|^2\delta_{AB}u_{,C} \\
&\quad +\delta_{AC}u_{,p}R_{PB}-\delta_{AB}u_{,p}R_{PC}+(n-2)\delta_{AB}u_{,p}u_{,pC}-(n-2)\delta_{AC}u_{,p}u_{,pB} \\
&\quad +(n-2)\delta_{AC}|Du|^2u_{,B}-(n-2)\delta_{AB}|Du|^2u_{,C}+\Delta u\delta_{AB}u_{,C}-\Delta u\delta_{AC}u_{,B} \\
&\quad +(n-2)|Du|^2\delta_{AB}u_{,C}-(n-2)|Du|^2\delta_{AC}u_{,B}-\frac{1}{n-1}\delta_{AC}u_{,B}R+\frac{1}{n-1}\delta_{AB}u_{,C}R \\
&\quad -2\delta_{AB}u_{,C}\Delta u+2\delta_{AC}u_{,B}\Delta u-(n-2)\delta_{AB}u_{,C}|Du|^2+(n-2)\delta_{AC}u_{,B}|Du|^2 \\
&\quad -\frac{1}{2(n-1)}(\delta_{AB}R_{,C}-\delta_{AC}R_{,B})-\delta_{AC}(\Delta u)_{,B}+\delta_{AB}(\Delta u)_{,C} \\
&\quad -(n-2)\delta_{AC}u_{,p}u_{,pB}+(n-2)\delta_{AB}u_{,p}u_{,pC}\} \quad (2-99) \\
&= \frac{1}{e^{3u}}\{R_{AB,C}-R_{AC,B}-\frac{1}{2(n-1)}(\delta_{AB}R_{,C}-\delta_{AC}R_{,B})
\end{aligned}$$

$$\begin{aligned}
& -(n-2)u_{,P} \left[ -\frac{1}{n-2}(R_{AB}\delta_{PC} - R_{AC}\delta_{PB} + \delta_{AB}R_{PC} - \delta_{AC}R_{PB}) \right. \\
& + \frac{R}{(n-1)(n-2)}(\delta_{AB}\delta_{PC} - \delta_{AC}\delta_{PB}) + W_{APBC} \left. \right] + R_{AC}u_{,B} - R_{AB}u_{,C} \\
& + \delta_{AC}u_{,P}R_{PB} - \delta_{AB}u_{,P}R_{PC} + \frac{R}{n-1}\delta_{AB}u_{,C} - \frac{R}{n-1}\delta_{AC}u_{,B} \}
\end{aligned} \quad (2-100)$$

$$= \frac{1}{e^{3u}} \{ Bak - (n-2)u_{,P}W_{APBC} \}. \quad (2-101)$$

当  $n=3$  时,  $W=0$ , 所以

$$\widetilde{Bak} = \frac{1}{e^{3u}} Bak.$$

这样, 用活动标架法证明了下面的定理.

**定理 2.5:** 在  $(M, ds^2)$ ,  $(M, ds^2)$  的标架  $\tilde{S}$ ,  $S$  下,  $U, V, W, Bak$ , Schouten 张量的变化规律为

•  $\dim_R N = 2$

$$\tilde{U} = \frac{1}{e^{2u}} (U + \frac{1}{2} \Delta u \delta \otimes \delta), \quad (2-102)$$

$$\tilde{V} = 0, \quad \tilde{W} = 0, \quad (2-103)$$

$$\tilde{X} = \frac{1}{e^{2u}} (X + \frac{1}{2} \Delta u \delta \otimes \delta). \quad (2-104)$$

•  $\dim_R N = 3$

$$\tilde{U} = \frac{1}{e^{2u}} (U + \frac{1}{3} \Delta u \delta \otimes \delta + \frac{1}{6} |Du|^2 \delta \otimes \delta), \quad (2-105)$$

$$\tilde{V} = \frac{1}{e^{2u}} [V + \frac{1}{3} (|Du|^2 - \Delta u) \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta], \quad (2-105)$$

$$\tilde{W} = 0, \quad \tilde{X} = \frac{1}{e^{2u}} [X + \frac{1}{2} |Du|^2 \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta], \quad (2-106)$$

$$\widetilde{Scht} = \frac{1}{e^{2u}} [Scht - (D^2 u - Du \otimes Du) - \frac{1}{2} |Du|^2 \delta], \quad (2-107)$$

$$\widetilde{Bak} = \frac{1}{e^{3u}} Bak. \quad (2-109)$$

•  $\dim_R N \geq 4$

$$\tilde{U} = \frac{1}{e^{2u}} [U + \frac{1}{n} \Delta u \delta \otimes \delta + \frac{n-2}{2n} |Du|^2 \delta \otimes \delta], \quad (2-110)$$

$$\tilde{V} = \frac{1}{e^{2u}} [V + \frac{1}{n} (|Du|^2 - \Delta u) \delta \otimes \delta + (D^2 u - Du \otimes Du) \otimes \delta], \quad (2-111)$$

$$\tilde{X} = \frac{1}{e^{2u}} \left[ X + \frac{1}{2} |Du|^2 \delta \otimes \delta + (D^2u - Du \otimes Du) \otimes \delta \right], \quad (2-112)$$

$$\tilde{W} = \widetilde{Riem} - \tilde{X} = \frac{1}{e^{2u}} W, \quad (2-113)$$

$$\widetilde{Sch} = \frac{1}{e^{2u}} \left[ Sch - (n-2)(D^2u - Du \otimes Du) - \frac{n-2}{2} |Du|^2 \delta \right]. \quad (2-114)$$

在共形几何中, 下面的定理是众所周知的.

**定理 2.6** (参见文献[102]): 设  $(N, ds^2)$  是局部共形平坦流形当且仅当满足下列一种情形.

- $\dim_R N = 2$  时: 总是局部共形平坦流形;
- $\dim_R N = 3$  时:  $Bak = 0$ , 即 Schouten 张量的一阶导数指标可交换;
- $\dim_R N \geq 4$  时:  $Weyl = 0$ , 即黎曼张量由 Schouten 张量决定.

特别的, 完备、单连通、常截面曲率  $c$  的空间——空间形式总是局部共形平坦流形, 记为  $R^n(c)$ , 满足以下简单关系:

$$R_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad (2-115)$$

$$R_{AB} = (n-1)c\delta_{AB}, \quad R = n(n-1)c. \quad (2-116)$$

## 第3章 子流形基本理论

本章主要研究子流形的基本方程,包括:结构方程、变分公式。同时也列出了很多子流形的例子。大部分内容都是新的。本章的指标采用如下两个约定。1. 爱因斯坦约定:重复指标表示求和;2. 指标范围:

$$1 \leq A, B, C, D, \dots \leq n+p, 1 \leq i, j, k, l, \dots \leq n, n+1 \leq \alpha, \beta, \gamma, \delta, \dots \leq n+p.$$

### 3.1 子流形结构方程

本节主要研究子流形的结构方程。参见文献[2,4]。

设  $x:(M^n, ds^2) \rightarrow (N^{n+p}, d\bar{s}^2)$  是子流形,  $x$  是等距浸入, 即是  $x^* d\bar{s}^2 = ds^2$ 。设  $S = (S_I, S_A)^i$  是  $TN$  的局部正交标架, 对偶地, 设  $\sigma = (\sigma^I, \sigma^A)$  是  $T^*N$  的局部正交标架。那么  $e = x^* S = x^*(S_I, S_A) = (e_I, e_A)$  是  $M$  上的拉回向量丛  $x^* TN = TM \oplus T^\perp M$  局部正交标架, 对偶地,  $\theta^I = x^* \sigma^I$  是  $T^*M$  的局部正交标架。

在子流形几何中一个基本的重要事实是

$$\theta^A = x^* \sigma^A = 0.$$

有等式

$$ds^2 = \sum_i (\theta^i)^2, d\bar{s}^2 = \sum_A (\sigma^A)^2.$$

设  $\omega, \Omega$  是  $TN$  上的联络和曲率形式, 在不致混淆的情况下, 设  $D$  是联络。

$$x^* \omega = \phi, x^* \Omega = \Phi, \quad (3-1)$$

$$DS = \omega S = D \begin{pmatrix} S_I \\ S_A \end{pmatrix} = \begin{pmatrix} \omega_I^I & \omega_I^A \\ \omega_A^I & \omega_A^A \end{pmatrix} \times \begin{pmatrix} S_I \\ S_A \end{pmatrix}, \quad (3-2)$$

$$D^2 S = \Omega S = D^2 \begin{pmatrix} S_I \\ S_A \end{pmatrix} = \begin{pmatrix} \Omega_I^I & \Omega_I^A \\ \Omega_A^I & \Omega_A^A \end{pmatrix} \times \begin{pmatrix} S_I \\ S_A \end{pmatrix}, \quad (3-3)$$

$$De = x^*(\omega)e = \phi e = D \begin{pmatrix} e_I \\ e_A \end{pmatrix} = \begin{pmatrix} \phi_I^I & \phi_I^A \\ \phi_A^I & \phi_A^A \end{pmatrix} \times \begin{pmatrix} e_I \\ e_A \end{pmatrix}, \quad (3-4)$$

$$D^2 e = x^*(\Omega)e = \Phi e = D^2 \begin{pmatrix} e_I \\ e_A \end{pmatrix} = \begin{pmatrix} \Phi_I^I & \Phi_I^A \\ \Phi_A^I & \Phi_A^A \end{pmatrix} \times \begin{pmatrix} e_I \\ e_A \end{pmatrix}. \quad (3-5)$$

从而  $(D, e_I, \phi_I^I)$  是  $TM$  的联络,  $(D, e_A, \phi_A^A)$  是  $T^\perp M$  的联络,  $\phi_i^A = h_{ij}^A \theta^j$  是  $M$  的第

二基本型, 记为

$$B = \sum_{ij\alpha} h_{ij}^{\alpha} \theta^i \otimes \theta^j \otimes e_{\alpha}, B_{ij} = \sum_{\alpha} h_{ij}^{\alpha} e_{\alpha}.$$

从第二基本型出发, 我们可以定义出很多的基本符号, 在下文之中反复应用。

当余维数为 1 时,  $p=1$ , 记

$$B = h_{ij} \theta^i \otimes \theta^j, A = (h_{ij})_{n \times n}, H = \frac{1}{n} \sum_i h_{ii}, S = \sum_{ij} (h_{ij})^2, \quad (3-6)$$

$$\hat{h}_{ij} = h_{ij} - H \delta_{ij}, \hat{B} = \hat{h}_{ij} \theta^i \otimes \theta^j = B - H ds^2, \quad (3-7)$$

$$\hat{A} = (\hat{h}_{ij})_{n \times n} = A - HI, \hat{S} = \sum_{ij} (\hat{h}_{ij})^2 = S - nH^2, \quad (3-8)$$

$$P_k = \text{tr}(A^k), P_1 = \sum_i h_{ii}, P_2 = \sum_{ij} (h_{ij})^2 = S, P_3 = \sum_{ijk} h_{ij} h_{jk} h_{ki}, \quad (3-9)$$

$$\dot{P}_k = \text{tr}(\dot{A}^k), \dot{P}_1 = 0, \dot{P}_2 = \sum_{ij} (\dot{h}_{ij})^2 = P_2 - \frac{1}{n} (P_1)^2 = \hat{S}, \dot{P}_3 = \sum_{ijk} \dot{h}_{ij} \dot{h}_{jk} \dot{h}_{ki}, \quad (3-10)$$

$$\rho = S - nH^2 = \hat{S}. \quad (3-11)$$

当余维数大于 1 时,  $p \geq 2$ , 记

$$B = h_{ij}^{\alpha} \theta^i \otimes \theta^j \otimes e_{\alpha}, B_{ij} = \sum_{\alpha} h_{ij}^{\alpha} e_{\alpha}, A_{\alpha} = A^{\alpha} = (h_{ij}^{\alpha})_{n \times n}, \quad (3-12)$$

$$H^{\alpha} = \frac{1}{n} \sum_i h_{ii}^{\alpha}, \vec{H} = \sum_{\alpha} H^{\alpha} e_{\alpha}, H = \sqrt{\sum_{\alpha} (H^{\alpha})^2}, S = \sum_{ij\alpha} (h_{ij}^{\alpha})^2, \quad (3-13)$$

$$\hat{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha} \delta_{ij}, \hat{B} = \hat{h}_{ij}^{\alpha} \theta^i \otimes \theta^j \otimes e_{\alpha} = B - \vec{H} \otimes ds^2, \quad (3-14)$$

$$\hat{B}_{ij} = \hat{h}_{ij}^{\alpha} \otimes e_{\alpha} = \sum_{\alpha} (h_{ij}^{\alpha} - H^{\alpha} \delta_{ij}) e_{\alpha} = B_{ij} - \vec{H} \delta_{ij}, \quad (3-15)$$

$$\hat{A}_{\alpha} = \hat{A}^{\alpha} = (\hat{h}_{ij}^{\alpha})_{n \times n} = (h_{ij}^{\alpha} - H^{\alpha} \delta_{ij}) = A_{\alpha} - H^{\alpha} I = A^{\alpha} - H^{\alpha} I, \quad (3-16)$$

$$S_{\alpha\beta} = \text{tr}(A_{\alpha} A_{\beta}) = \sum_{ij} h_{ij}^{\alpha} h_{ij}^{\beta}, S = \sum_{\alpha} S_{\alpha\alpha}, \quad (3-17)$$

$$S_{\alpha\beta\gamma} = \text{tr}(A_{\alpha} A_{\beta} A_{\gamma}) = \sum_{ijk} h_{ij}^{\alpha} h_{jk}^{\beta} h_{ki}^{\gamma}, \quad (3-18)$$

$$S_{\alpha\beta\gamma\delta} = \text{tr}(A_{\alpha} A_{\beta} A_{\gamma} A_{\delta}) = \sum_{ijkl} h_{ij}^{\alpha} h_{jk}^{\beta} h_{kl}^{\gamma} h_{li}^{\delta}, \quad (3-19)$$

$$N(A_{\alpha}) = \text{tr}(A_{\alpha} A_{\alpha}^t) = \sum_{ij} (h_{ij}^{\alpha})^2 = S_{\alpha\alpha}, \quad (3-20)$$

$$N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) = \text{tr}[(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha})(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha})^t] = 2(S_{\alpha\alpha\beta\beta} - S_{\alpha\beta\alpha\beta}), \quad (3-21)$$

$$\hat{S} = \sum_{ij\alpha} (\hat{h}_{ij}^{\alpha})^2 = \sum_{ij\alpha} (h_{ij}^{\alpha} - H^{\alpha} \delta_{ij})^2 = S - nH^2, \quad (3-22)$$

$$\hat{S}_{\alpha\beta} = \text{tr}(\hat{A}_{\alpha} \hat{A}_{\beta}) = \text{tr}[(A_{\alpha} - H^{\alpha} I)(A_{\beta} - H^{\beta} I)] = S_{\alpha\beta} - nH^{\alpha} H^{\beta}, \quad (3-23)$$

$$\hat{S}_{\alpha\beta\gamma} = \text{tr}(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}) = \text{tr}[(A_{\alpha} - H^{\alpha} I)(A_{\beta} - H^{\beta} I)(A_{\gamma} - H^{\gamma} I)]$$

$$= S_{\alpha\beta\gamma} + 2nH^\alpha H^\beta H^\gamma - S_{\alpha\beta} H^\gamma - S_{\alpha\gamma} H^\beta - S_{\beta\gamma} H^\alpha, \quad (3-24)$$

$$\begin{aligned} \hat{S}_{\alpha\beta\gamma\delta} &= \text{tr}(\hat{A}_\alpha \hat{A}_\beta \hat{A}_\gamma \hat{A}_\delta) = \text{tr}[(A_\alpha - H^\alpha I)(A_\beta - H^\beta I)(A_\gamma - H^\gamma I)(A_\delta - H^\delta I)] \\ &= S_{\alpha\beta\gamma\delta} - 3nH^\alpha H^\beta H^\gamma H^\delta - H^\alpha S_{\beta\gamma\delta} - H^\beta S_{\alpha\gamma\delta} - H^\gamma S_{\alpha\beta\delta} - H^\delta S_{\alpha\beta\gamma} + S_{\alpha\beta} H^\gamma H^\delta \\ &\quad + S_{\alpha\gamma} H^\beta H^\delta + S_{\alpha\delta} H^\beta H^\gamma + S_{\beta\gamma} H^\alpha H^\delta \\ &\quad + S_{\beta\delta} H^\alpha H^\gamma + S_{\gamma\delta} H^\alpha H^\beta, \end{aligned} \quad (3-25)$$

$$N(\hat{A}_\alpha) = \text{tr}(\hat{A}_\alpha \hat{A}_\alpha^t) = \sum_j (\hat{h}_{ij}^\alpha)^2 = \hat{S}_{\alpha\alpha}, \quad (3-26)$$

$$N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) = \text{tr}[(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha)(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha)^t] \quad (3-27)$$

$$= 2(\hat{S}_{\alpha\alpha\beta\beta} - \hat{S}_{\alpha\beta\alpha\beta}), \quad (3-28)$$

$$\rho = S - nH^2 = \sum_\alpha S_{\alpha\alpha} - n(H^\alpha)^2 = \sum_\alpha \hat{S}_{\alpha\alpha} = \hat{S}. \quad (3-29)$$

显然, 有

$$S_{\alpha\alpha} \geq 0, S \geq 0, \hat{S}_{\alpha\alpha} \geq 0, \hat{S} \geq 0, \rho \geq 0. \quad (3-30)$$

记  $TN$ ,  $TM$ ,  $T^\perp M$  上的微分算子、Christoffel 和黎曼曲率符号分别为  $d$ ,  $d_M$ ;  $\bar{\Gamma}_{AC}^B$ ,  $\omega$ ,  $\bar{R}_{ABCD}$ ,  $\Omega$ ;  $\Gamma_{ik}^j$ ,  $\phi_i^j$ ,  $R_{ijkl}$ ,  $\Omega^\tau$ ;  $\Gamma_{\alpha i}^\beta$ ,  $\phi_\alpha^\beta$ ,  $R_{\alpha\beta ij}^\perp$ ,  $\Omega^\perp$ .

于是

$$\omega + \omega' = 0, d\sigma - \sigma \wedge \omega = 0, \quad (3-31)$$

$$\Omega + \Omega' = 0, d\omega - \omega \wedge \omega = \Omega, \quad (3-32)$$

$$\sigma \wedge \Omega = 0, \sigma \wedge \Omega' = 0, d\Omega = \omega \wedge \Omega - \Omega \wedge \omega. \quad (3-33)$$

对(3-31)拉回

$$\phi + \phi' = 0, \phi_A^B = \Gamma_{Ai}^B \theta^i, x^* \bar{\Gamma}_{Ai}^B = \Gamma_{Ai}^B, \quad (3-34)$$

$$\Gamma_{ik}^j = -\Gamma_{jk}^i, \Gamma_{ij}^\alpha = -\Gamma_{\alpha j}^i =: h_{ij}^\alpha, \Gamma_{\alpha i}^\beta = -\Gamma_{\beta i}^\alpha, \quad (3-35)$$

$$d_M \theta - \theta \wedge \phi = 0, d_M \theta^j - \theta^j \wedge \phi_i^j - \theta^j \wedge \phi_A^j = d_M \theta^j - \theta^j \wedge \phi_i^j = 0, \quad (3-36)$$

$$d_M \theta^A - \theta^j \wedge \phi_j^A - \theta^A \wedge \phi_A^A = -\theta^j \wedge \phi_j^A = 0, h_{ij}^\alpha = h_{ji}^\alpha. \quad (3-37)$$

对(3-32)拉回

$$\Phi + \Phi' = 0, \Phi_{AB} = \frac{1}{2} x^* (\bar{R}_{ABij}) \theta^i \wedge \theta^j, \quad (3-38)$$

$$R_{ijkl} = x^* \bar{R}_{ijkl} = -R_{jikl} = -R_{ijlk}, \quad (3-39)$$

$$R_{\alpha\beta ij}^\perp = x^* \bar{R}_{\alpha\beta ij}^\perp = -R_{\beta\alpha ij}^\perp = -R_{\alpha\beta ji}^\perp. \quad (3-40)$$

对矩阵  $\Phi = d_M \phi - \phi \wedge \phi$  的第一部分

$$\Phi_{II} = \Omega^I - \phi_i^A \wedge \phi_A^I, \quad (3-41)$$

$$\frac{1}{2} \bar{R}_{ijkl} \theta^k \wedge \theta^l = \frac{1}{2} R_{ijkl} \theta^k \wedge \theta^l + \sum_\alpha h_{ik}^\alpha h_{jl}^\alpha \theta^k \wedge \theta^l \quad (3-42)$$

$$= \frac{1}{2} (R_{ijkl} + \sum_\alpha h_{ik}^\alpha h_{jl}^\alpha - h_{jk}^\alpha h_{il}^\alpha) \theta^k \wedge \theta^l, \quad (3-43)$$



$$\bar{R}_{ijkl} = R_{ijkl} + \sum_{\alpha} h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{jk}^{\alpha} h_{il}^{\alpha}. \quad (3-44)$$

对矩阵  $\Phi = d_M \phi - \phi \wedge \phi$  的第二部分

$$\Phi_l^A = d_M \phi_l^A - \phi_l^i \wedge \phi_i^A - \phi_l^i \wedge \phi_i^A, \quad (3-45)$$

$$\frac{1}{2} \bar{R}_{ijk}^{\alpha} \theta^i \wedge \theta^k = \Phi_i^{\alpha} = d_M h_{ik}^{\alpha} \wedge \theta^k - h_{ip} \phi_k^p \wedge \theta^k - h_{pk}^{\alpha} \phi_i^p \wedge \theta^k + h_{ik}^{\beta} \phi_p^{\alpha} \theta^k \quad (3-46)$$

$$= \frac{1}{2} (h_{ik,j}^{\alpha} - h_{ij,k}^{\alpha}) \theta^j \wedge \theta^k, \quad (3-47)$$

$$\bar{R}_{ijk}^{\alpha} = h_{ik,j}^{\alpha} - h_{ij,k}^{\alpha}. \quad (3-48)$$

对矩阵  $\Phi = d_M \phi - \phi \wedge \phi$  的第四部分

$$\Phi_{AA} = d_M \phi_A^A - \phi_A^i \wedge \phi_i^A - \phi_A^i \wedge \phi_i^A = \Omega^{\perp} - \phi_A^i \wedge \phi_i^A, \quad (3-49)$$

$$\frac{1}{2} \bar{R}_{\alpha\beta ij} \theta^i \wedge \theta^j = \frac{1}{2} R_{\alpha\beta ij}^{\perp} \theta^i \wedge \theta^j + \sum_p h_{ip}^{\alpha} h_{pj}^{\beta} \theta^i \wedge \theta^j \quad (3-50)$$

$$= \frac{1}{2} (R_{\alpha\beta ij}^{\perp} + \sum_p h_{ip}^{\alpha} h_{pj}^{\beta} - h_{ip}^{\beta} h_{pj}^{\alpha}), \quad (3-51)$$

$$\bar{R}_{\alpha\beta ij} = R_{\alpha\beta ij}^{\perp} + \sum_p h_{ip}^{\alpha} h_{pj}^{\beta} - h_{ip}^{\beta} h_{pj}^{\alpha}. \quad (3-52)$$

对(3-33)拉回

$$\theta \wedge \Omega = 0, \theta \wedge \Omega' = 0, \theta' \wedge (\Omega'_i)' = 0, \theta' \wedge (\Omega'_i) = 0, \quad (3-53)$$

$$\frac{1}{2} \bar{R}_{ijkl} \theta^j \wedge \theta^k \wedge \theta^l = 0, \bar{R}_{ijkl} + \bar{R}_{iklj} + \bar{R}_{iljk} = 0, \quad (3-54)$$

$$\frac{1}{2} \bar{R}_{ijk}^{\alpha} \theta^i \wedge \theta^j \wedge \theta^k = 0, \bar{R}_{ijk}^{\alpha} + \bar{R}_{jki}^{\alpha} + \bar{R}_{kij}^{\alpha} = 0. \quad (3-55)$$

$$d_M \Phi = \phi \wedge \Phi - \Phi \wedge \phi, \quad (3-56)$$

对于矩阵  $d_M \Phi = \phi \wedge \Phi - \Phi \wedge \phi$  的第一部分

$$d_M \Phi_{ll} = \phi_l^i \wedge \Phi_i^l + \phi_l^i \wedge \Phi_i^l - \Phi_l^i \wedge \phi_i^l - \Phi_l^i \wedge \phi_i^l, \quad (3-57)$$

$$\frac{1}{2} d_M \bar{R}_{ykl} \theta^k \wedge \theta^l - \frac{1}{2} \bar{R}_{ypl} \phi_k^p \theta^k \wedge \theta^l - \frac{1}{2} \bar{R}_{ykl} \phi_l^p \theta^k \wedge \theta^l \quad (3-58)$$

$$- \frac{1}{2} \bar{R}_{pkl} \phi_i^p \theta^k \wedge \theta^l - \frac{1}{2} \bar{R}_{ipkl} \phi_j^p \theta^k \wedge \theta^l + \frac{1}{2} \sum_{\alpha} (h_{im}^{\alpha} \bar{R}_{jkl}^{\alpha} - h_{jm}^{\alpha} \bar{R}_{ikl}^{\alpha}) \theta^k \wedge \theta^l \wedge \theta^m = 0, \quad (3-59)$$

$$\frac{1}{2} (\bar{R}_{ijkl,m} - \sum_{\alpha} (\bar{R}_{ikl}^{\alpha} h_{jm}^{\alpha} - \bar{R}_{jkl}^{\alpha} h_{im}^{\alpha})) \theta^k \wedge \theta^l \wedge \theta^m = 0, \quad (3-60)$$

$$\begin{aligned} & \bar{R}_{ijkl,m} + \bar{R}_{ijlm,k} + \bar{R}_{ijmk,l} - \sum_{\alpha} (\bar{R}_{ikl}^{\alpha} h_{jm}^{\alpha} - \bar{R}_{jkl}^{\alpha} h_{im}^{\alpha}) - \sum_{\alpha} (\bar{R}_{ilm}^{\alpha} h_{jk}^{\alpha} - \bar{R}_{jlm}^{\alpha} h_{ik}^{\alpha}) \\ & - \sum_{\alpha} (\bar{R}_{imk}^{\alpha} h_{jl}^{\alpha} - \bar{R}_{jmk}^{\alpha} h_{il}^{\alpha}) = 0. \end{aligned} \quad (3-61)$$

对于矩阵(3-56)的第二部分

$$d_M \Phi_I^A = \phi_I' \wedge \Phi_I^A + \phi_I^A \wedge \Phi_A^A - \Phi_I' \wedge \phi_I^A - \Phi_I^A \wedge \phi_A^A, \quad (3-62)$$

$$\frac{1}{2}(\bar{R}_{ijk,l} + (\sum_p \bar{R}_{ipjk} h_{pl}^\alpha - \sum_\beta \bar{R}_{\beta jk} h_{il}^\beta)) \theta^i \wedge \theta^k \wedge \theta^l = 0, \quad (3-63)$$

$$\begin{aligned} & \bar{R}_{ijk,l}^\alpha + \bar{R}_{ikl,j}^\alpha + \bar{R}_{ilj,k}^\alpha + (\sum_p \bar{R}_{ipjk} h_{pl}^\alpha - \sum_\beta \bar{R}_{\beta jk} h_{il}^\beta) + \\ & (\sum_p \bar{R}_{ipkl} h_{pj}^\alpha - \sum_\beta \bar{R}_{\beta kl} h_{ij}^\beta) + (\sum_p \bar{R}_{iplj} h_{pk}^\alpha - \sum_\beta \bar{R}_{\beta lj} h_{ik}^\beta) = 0. \end{aligned} \quad (3-64)$$

对于矩阵(3-56)的第四部分

$$d_M \Phi_A^A = \phi_A' \wedge \Phi_I^A + \phi_A^A \wedge \Phi_A^A - \Phi_A' \wedge \phi_I^A - \Phi_A^A \wedge \phi_A^A, \quad (3-65)$$

$$\frac{1}{2}[\bar{R}_{\alpha\beta ij,k} - \sum_p (\bar{R}_{p\beta ij} h_{pk}^\beta - \bar{R}_{p\beta j} h_{pk}^\alpha)] \theta^i \wedge \theta^j \wedge \theta^k = 0, \quad (3-66)$$

$$\begin{aligned} & \bar{R}_{\alpha\beta ij,k} + \bar{R}_{\alpha\beta ki,j} + \bar{R}_{\alpha\beta ki,i} - \sum_p (\bar{R}_{p\beta ij} h_{pk}^\beta - \bar{R}_{p\beta j} h_{pk}^\alpha) - \\ & \sum_p (\bar{R}_{p\beta jk} h_{pi}^\beta - \bar{R}_{p\beta jk} h_{pi}^\alpha) - \sum_p (\bar{R}_{pki} h_{pj}^\beta - \bar{R}_{pki} h_{pj}^\alpha) = 0. \end{aligned} \quad (3-67)$$

对于第二基本型, 下面的 Ricci 恒等式是重要的

$$h_{ij,kl}^\alpha - h_{ij,lk}^\alpha = \sum_p h_{pj}^\alpha R_{ipkl} + \sum_p h_{ip}^\alpha R_{jpkl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}^\perp \quad (3-68)$$

$$\begin{aligned} = & \sum_p h_{pj}^\alpha [\bar{R}_{ipkl} - \sum_\beta (h_{ik}^\beta h_{pl}^\beta - h_{il}^\beta h_{pk}^\beta)] + \sum_p h_{ip}^\alpha [\bar{R}_{jpkl} - \sum_\beta (h_{jk}^\beta h_{pl}^\beta - h_{jl}^\beta h_{pk}^\beta)] \\ & + \sum_\beta h_{ij}^\beta [\bar{R}_{\alpha\beta kl} - \sum_p (h_{kp}^\alpha h_{pl}^\beta - h_{lp}^\alpha h_{pk}^\beta)] \end{aligned} \quad (3-69)$$

$$\begin{aligned} = & \sum_p h_{pj}^\alpha \bar{R}_{ipkl} + \sum_p h_{ip}^\alpha \bar{R}_{jpkl} + \sum_\beta h_{ij}^\beta \bar{R}_{\alpha\beta kl} + \\ & \sum_{p\beta} (h_{il}^\beta h_{jp}^\alpha h_{pk}^\beta - h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta) + \sum_{p\beta} (h_{ip}^\alpha h_{pk}^\beta h_{jl}^\beta - h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) + \\ & \sum_{p\beta} (h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta - h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta). \end{aligned} \quad (3-70)$$

综上所述, 我们得到了子流形的结构方程定理。

**定理 3.1** (参见文献[2]): 设  $x: M \rightarrow N$  是子流形, 张量的变化规律为

$$h_{ij}^\alpha = h_{ji}^\alpha, \bar{R}_{ijkl} = R_{ijkl} + \sum_\alpha h_{ik}^\alpha h_{jl}^\alpha - h_{jk}^\alpha h_{il}^\alpha, \quad (3-71)$$

$$\bar{R}_{ijk}^\alpha = h_{ik,j}^\alpha - h_{ij,k}^\alpha, \bar{R}_{\alpha\beta ij} = R_{\alpha\beta ij}^\perp + \sum_p h_{ip}^\alpha h_{pj}^\beta - h_{ip}^\beta h_{pj}^\alpha, \quad (3-72)$$

$$R_{ij} = \sum_p \bar{R}_{ippj} - \sum_{p\alpha} h_{ip}^\alpha h_{pj}^\alpha + \sum_\alpha n H^\alpha h_{ij}^\alpha, \quad (3-73)$$

$$R = \sum_{ij} \bar{R}_{ijji} - S + n^2 H^2, \quad (3-74)$$

$$\begin{aligned} & \bar{R}_{ijk,l}^\alpha + \bar{R}_{ilm,j}^\alpha + \bar{R}_{ijm,k}^\alpha - \sum_\alpha (\bar{R}_{ikl}^\alpha h_{jm}^\alpha - \bar{R}_{jkl}^\alpha h_{im}^\alpha) \\ & - \sum_\alpha (\bar{R}_{ilm}^\alpha h_{jk}^\alpha - \bar{R}_{jlm}^\alpha h_{ik}^\alpha) - \sum_\alpha (\bar{R}_{imk}^\alpha h_{jl}^\alpha - \bar{R}_{jmk}^\alpha h_{il}^\alpha) = 0, \end{aligned} \quad (3-75)$$

$$\bar{R}_{ijk,l}^\alpha + \bar{R}_{ikl,j}^\alpha + \bar{R}_{ilj,k}^\alpha + (\sum_p \bar{R}_{ipjk} h_{pl}^\alpha - \sum_\beta \bar{R}_{\beta jk} h_{il}^\beta)$$

$$+ \left( \sum_p \bar{R}_{ipkl} h_{pj}^\alpha - \sum_\beta \bar{R}_{\beta ki} h_{ij}^\beta \right) + \left( \sum_p \bar{R}_{iplj} h_{pk}^\alpha - \sum_\beta \bar{R}_{\beta lj} h_{ik}^\beta \right) = 0, \quad (3-76)$$

$$\begin{aligned} & \bar{R}_{\alpha\beta ij, k} + \bar{R}_{\alpha\beta ik, i} + \bar{R}_{\alpha\beta ki, j} - \sum_p (\bar{R}_{pij} h_{pk}^\beta - \bar{R}_{pij} h_{pk}^\alpha) \\ & - \sum_p (\bar{R}_{pj k} h_{pi}^\beta - \bar{R}_{pj k} h_{pi}^\alpha) - \sum_p (\bar{R}_{pki} h_{pj}^\beta - \bar{R}_{pki} h_{pj}^\alpha) = 0. \end{aligned} \quad (3-77)$$

**注释 3.1:** 在定理 3.1 中, 前三行等式是经典的结果, 后面的 Bianchi 等式是新推导的结果, 当然也可以由 Gauss、Codazzi、Ricci 等式协变导数得到。

设  $N$  是空间形式  $R^{n+p}(c)$ , 众所周知有如下关系

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

代入定理 3.1, 可得如下推论:

**推论 3.1**(参见文献[2]): 设  $N$  是空间形式  $R^{n+p}(c)$ , 子流形  $x: M \rightarrow R^{n+p}(c)$  有以下结构方程。

$$dx = \theta^i e_i, de_i = \phi_i^j e_j + \phi_i^\alpha e_\alpha - c\theta^i x, de_\alpha = \phi_\alpha^i e_i + \phi_\alpha^\beta e_\beta, \quad (3-78)$$

$$h_{ij}^\alpha = h_{ji}^\alpha, R_{ijkl} = -c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{jk}^\alpha h_{il}^\alpha), \quad (3-79)$$

$$R_{ij} = c(n-1)\delta_{ij} - \sum_{p\alpha} h_{ip}^\alpha h_{pj}^\alpha + \sum_\alpha nH^\alpha h_{ij}^\alpha, \quad (3-80)$$

$$R = cn(n-1) - S + n^2 H^2, \quad (3-81)$$

$$h_{ik, j}^\alpha = h_{ij, k}^\alpha, R_{\alpha\beta ij}^\perp = - \sum_p (h_{ip}^\alpha h_{pj}^\beta - h_{ip}^\beta h_{pj}^\alpha), \quad (3-82)$$

$$\bar{R}_{ijkl, m} + \bar{R}_{ijlm, k} + \bar{R}_{ijmk, l} = 0, \bar{R}_{ijk, l}^\alpha + \bar{R}_{ikl, j}^\alpha + \bar{R}_{ijl, k}^\alpha = 0, \quad (3-83)$$

$$\bar{R}_{\alpha\beta ij, k} + \bar{R}_{\alpha\beta ik, i} + \bar{R}_{\alpha\beta ki, j} = 0. \quad (3-84)$$

**注释 3.2:** 在定理 3.1 和推论 3.1 之中, 对  $\bar{R}_{ABCD}$  的协变导数都是在拉回丛上进行的。

**命题 3.1:** 设  $x: M \rightarrow N$  是子流形, 有如下 Ricci 恒等式

- $p \geq 2$ , 一般子流形

$$\begin{aligned} h_{ij, kl}^\alpha - h_{ij, lk}^\alpha &= \sum_p h_{pj}^\alpha \bar{R}_{ipkl} + \sum_p h_{ip}^\alpha \bar{R}_{jpkl} + \sum_\beta h_{ij}^\beta \bar{R}_{\alpha\beta kl} \\ &+ \sum_{p\beta} (h_{il}^\beta h_{jp}^\alpha h_{pk}^\beta - h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta) + \sum_{p\beta} (h_{ip}^\alpha h_{pk}^\beta h_{jl}^\beta - h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) \\ &+ \sum_{p\beta} (h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta - h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta). \end{aligned} \quad (3-85)$$

- $p \geq 2$ , 空间形式中子流形

$$\begin{aligned} h_{ij, kl}^\alpha - h_{ij, lk}^\alpha &= c(\delta_{il} h_{jk}^\alpha - \delta_{ik} h_{jl}^\alpha + \delta_{jl} h_{ik}^\alpha - \delta_{jk} h_{il}^\alpha) \\ &+ \sum_{p\beta} (h_{il}^\beta h_{jp}^\alpha h_{pk}^\beta - h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta) + \sum_{p\beta} (h_{ip}^\alpha h_{pk}^\beta h_{jl}^\beta - h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) \\ &+ \sum_{p\beta} (h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta - h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta). \end{aligned} \quad (3-86)$$

- $p=1$ , 一般超曲面

$$h_{ij,kl} - h_{ij,kl} = \sum_p h_{pj} \bar{R}_{ipkl} + \sum_p h_{ip} \bar{R}_{jpkl} + \sum_p (h_{il} h_{jp} h_{pk} - h_{ik} h_{jp} h_{pl} + h_{ip} h_{pk} h_{jl} - h_{ip} h_{pl} h_{jk}). \quad (3-87)$$

•  $p=1$ , 空间形式中超曲面

$$h_{ij,kl} - h_{ij,kl} = c(\delta_{il} h_{jk} - \delta_{ik} h_{jl} + \delta_{jl} h_{ik} - \delta_{jk} h_{il}) + \sum_p (h_{il} h_{jp} h_{pk} - h_{ik} h_{jp} h_{pl} + h_{ip} h_{pk} h_{jl} - h_{ip} h_{pl} h_{jk}). \quad (3-88)$$

## 3.2 子流形共形变换

本节主要讨论子流形的共形变换, 沿用第2章和第3.1节的符号.

设  $\bar{u}: N \rightarrow R$  是光滑函数, 则  $\widetilde{d\bar{s}^2} = e^{2\bar{u}} d\bar{s}^2$  是  $d\bar{s}^2$  的共形变换,  $(N, \widetilde{d\bar{s}^2})$  的局部正交标架分别为

$$\tilde{S} = \frac{1}{e^{\bar{u}}} S = \frac{1}{e^{\bar{u}}} (S_I, S_A)^t, \quad (3-89)$$

$$\tilde{\sigma} = e^{\bar{u}} \sigma = e^{\bar{u}} (\sigma^I, \sigma^A). \quad (3-90)$$

记

$$d\bar{u} = \bar{u}_{,A} \sigma^A, \quad d\bar{u}_{,A} - \bar{u}_{,B} \omega_A^B = \bar{u}_{,AB} \sigma^B, \quad (3-91)$$

$$d\bar{u}_{,AB} - \bar{u}_{,CB} \omega_A^C - \bar{u}_{,AC} \omega_B^C = \bar{u}_{,ABC} \sigma^C, \quad (3-92)$$

$$D\bar{u} = (\bar{u}_{,A})_{1 \times (n+p)}, \quad D^2\bar{u} = (\bar{u}_{,AB})_{(n+p) \times (n+p)}, \quad D^3(\bar{u}) = (\bar{u}_{,ABC}), \quad (3-93)$$

$$x^* \bar{u} = u, \quad x^* \bar{u}_{,\alpha} = u_{,\alpha}, \quad x^* \bar{u}_{,i} = u_{,i}, \quad (3-94)$$

$$d_M u = \sum_i u_{,i} \theta^i, \quad d_M u_{,i} - \sum_p u_{,p} \phi_i^p = u_{,ij} \theta^j, \quad (3-95)$$

$$d_M u_{,\alpha} - u_{,\beta} \phi_\alpha^\beta = u_{,\alpha\beta} \theta^\beta, \quad (3-96)$$

$$Du = (u_{,i})_{1 \times n}, \quad D^2 u = (u_{,ij})_{n \times n}, \quad DD^\perp u = (u_{,\alpha i})_{p \times n}, \quad (3-97)$$

$$x^* \bar{u}_{,ij} = u_{,ij} - \sum_\alpha u_{,\alpha} h_{ij}^\alpha, \quad x^* \bar{u}_{,\alpha i} = u_{,\alpha i} + \sum_j h_{ij}^\alpha u_{,j}. \quad (3-98)$$

显然,  $\widetilde{ds^2} = e^{2u} ds^2$  是  $ds^2$  的共形变换, 因此子流形

$$x: (M, ds^2) \rightarrow (N, d\bar{s}^2), \quad x: (M, \widetilde{ds^2}) \rightarrow (N, \widetilde{d\bar{s}^2})$$

的局部正交标架是

$$\tilde{e} = x^* \tilde{S} = \frac{1}{e^u} e = \frac{1}{e^u} (e_I, e_A)^t, \quad (3-99)$$

$$\tilde{\theta} = x^* \tilde{\sigma} = e^u \theta = e^u (\theta^I, \theta^A). \quad (3-100)$$

由定理2.3, 可得到以下的定理。

**定理 3.2:** 在  $(M, \widetilde{ds}^2)$  和  $(M, ds^2)$  分别的标架  $\widetilde{e}, e$  下, 第二基本型的变换规律为

$$h_{ij}^\alpha = \frac{h_{ij}^\alpha - u_{,\alpha} \delta_{ij}}{e^u},$$

$$\widetilde{B} = \widetilde{h}_{ij}^\alpha \widetilde{\theta}^i \otimes \widetilde{\theta}^j \otimes \widetilde{e}_\alpha = B - \sum_\alpha u_{,\alpha} e_\alpha ds^2. \quad (3-101)$$

### 3.3 子流形的例子

**例 3.1:** 全测地子流形  $B=0$ 。欧氏空间中的超平面, 球面中的赤道。

**例 3.2:** 欧氏空间  $E^{n+1}$  中的单位球面  $S^n(1)$ , 显然的  $k_1 = k_2 = \cdots = k_n = 1$ 。

**例 3.3** (参见文献[2]): 设  $0 < r < 1$ ,  $M: S^m(r) \times S^{n-m}(\sqrt{1-r^2}) \rightarrow S^{n+1}(1)$ 。计算如下:

$$S^m(r) = \{rx_1: |x_1| = 1\} E^{m+1}, S^{n-m}(\sqrt{1-r^2}) = \{\sqrt{1-r^2}x_2: |x_2| = 1\} E^{n-m+1},$$

$$M := \{x = (rx_1, \sqrt{1-r^2}x_2)\} S^{n+1}(1) E^{n+2},$$

$$ds^2 = (rdx_1)^2 + (\sqrt{1-r^2}dx_2)^2, e_{n+1} = (-\sqrt{1-r^2}x_1, rx_2),$$

$$h_{ij}^{n+1} \theta^i \otimes \theta^j =: h_{ij} \theta^i \otimes \theta^j = -\langle dx, de_{n+1} \rangle$$

$$= \frac{\sqrt{1-r^2}}{r} (rdx_1)^2 - \frac{r}{\sqrt{1-r^2}} (\sqrt{1-r^2}dx_2)^2,$$

$$k_1 = k_2 = \cdots = k_m = \frac{\sqrt{1-r^2}}{r}, k_{m+1} = k_{m+2} = \cdots = k_n = -\frac{r}{\sqrt{1-r^2}}.$$

**例 3.4** (参见文献[38]): 设  $0 < a_1, a_2, \cdots, a_{p+1} < 1$  满足  $\sum_{i=1}^{p+1} (a_i)^2 = 1$ , 设正整数  $n_1, n_2, \cdots, n_{p+1}$  满足  $\sum_{i=1}^{p+1} n_i = n$ .  $M := S^{n_1}(a_1) \times \cdots \times S^{n_{p+1}}(a_{p+1}) \rightarrow S^{n+p}(1)$ , 计算如下:

$$S^{n_1}(a_1) = \{a_1 x_1: |x_1| = 1\} E^{n_1+1}, \cdots,$$

$$S^{n_{p+1}}(a_{p+1}) = \{a_{p+1} x_{p+1}: |x_{p+1}| = 1\} E^{n_{p+1}+1},$$

$$M = \{x: x = (a_1 x_1, a_2 x_2, \cdots, a_{p+1} x_{p+1})\} \rightarrow S^{n+p}(1) E^{n+p+1},$$

$$ds^2 = \sum_{i=1}^{p+1} (a_i dx_i)^2, e_\alpha = (a_{\alpha 1} x_1, a_{\alpha 2} x_2, \cdots, a_{\alpha(p+1)} x_{p+1}), (n+1) \leq \alpha \leq (n+p),$$

$$h_{ij}^\alpha \theta^i \otimes \theta^j = -\langle dx, de_\alpha \rangle = -\sum_{i=1}^{p+1} \frac{a_{\alpha i}}{a_i} (a_i dx_i)^2,$$

$$(h_{ij}^a) = \begin{bmatrix} -\frac{a_{a1}}{a_1} E_{n_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\frac{a_{a(p+1)}}{a_{p+1}} E_{n_{p+1}} \end{bmatrix}; A = \begin{bmatrix} a_1 & \cdots & a_{p+1} \\ a_{(n+1)1} & \cdots & a_{(n+1)(p+1)} \\ \vdots & \vdots & \vdots \\ a_{(n+p)1} & \cdots & a_{(n+p)(p+1)} \end{bmatrix},$$

$$A^t A = I, \sum_{\alpha} a_{\alpha} a_{\alpha j} = \delta_{ij} - a_i a_j, \sum_i a_{\alpha} a_i = 0, \sum_i a_{\alpha} a_{\beta i} = \delta_{\alpha\beta}.$$

**例 3.5** (参见文献[80-92]): 设  $M$  是  $S^{n+1}(1)$  中的闭的等参超曲面, 设  $k_1 > k_2 > \cdots > k_g$  是常主曲率重数分别为  $m_1, m_2, \cdots, m_g, n = m_1 + m_2 + \cdots + m_g$ . 有:

1.  $g$  只能取 1, 2, 3, 4, 6;
2. 当  $g=1$  时,  $M$  是全脐;
3. 当  $g=2$  时,  $M = S^m(r) \times S^{n-m}(\sqrt{1-r^2})$ ;
4. 当  $g=3$  时,  $m_1 = m_2 = m_3 = 2^k, k=0, 1, 2, 3$ ;
5. 当  $g=4$  时,  $m_1 = m_3, m_2 = m_4, (m_1, m_2) = (2, 2)$  或  $(4, 5)$  或  $m_1 + m_2 + 1 \equiv 0 \pmod{2^{\phi(m_1-1)}}$ , 函数  $\phi(m) = \#\{s: 1 \leq s \leq m, s \equiv 0, 1, 2, 4 \pmod{8}\}$ ;
6. 当  $g=6$  时,  $m_1 = m_2 = \cdots = m_6 = 1$  或者 2;
7. 存在一个角度  $\theta, 0 < \theta < \frac{\pi}{g}$ , 使得  $k_{\alpha} = \cot(\theta + \frac{\alpha-1}{g}\pi), \alpha=1, 2, \cdots, g$ .

关于等参超曲面, 最近唐梓洲教授等有一系列的结果, 参见文献[85-89, 91, 92]。

**例 3.6** (参见文献[84]): Nomizu 等参超曲面, 令  $S^{n+1}(1) = \{(x_1, x_2, \cdots, x_{2r+1}, x_{2r+2}) \in R^{n+2} = R^{2r+2}: |x| = 1\}$ , 其中  $n=2r \geq 4$ . 定义函数:

$$F(x) = \left( \sum_{i=1}^{r+1} (x_{2i-1}^2 - x_{2i}^2) \right)^2 + 4 \left( \sum_{i=1}^{r+1} x_{2i-1} x_{2i} \right)^2.$$

考虑由函数  $F(x)$  定义的超曲面

$$M_t^r = \{x \in S^{n+1}: F(x) = \cos^2(2t)\}, 0 < t < \frac{\pi}{4}.$$

$M_t^r$  对固定参数  $t$  的主曲率为

$$k_1 = k_2 = \cdots = k_{r-1} = \cot(-t), k_r = \cot\left(\frac{\pi}{4} - t\right),$$

$$k_{r+1} = k_{r+2} = \cdots = k_{n-1} = \cot\left(\frac{\pi}{2} - t\right), k_n = \cot\left(\frac{3\pi}{4} - t\right).$$

**例 3.7** (参见文献[2]): Veronese 曲面. 设  $R^3$  和  $R^5$  的自然标架分别为

$$(x, y, z), (u_1, u_2, u_3, u_4, u_5).$$

定义映射如下:

$$\begin{aligned}
u_1 &= \frac{1}{\sqrt{3}}yz, \quad u_2 = \frac{1}{\sqrt{3}}xz, \quad u_3 = \frac{1}{\sqrt{3}}xy, \\
u_4 &= \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2), \\
x^2 + y^2 + z^2 &= 3.
\end{aligned}$$

该映射给出了一个嵌入  $i: RP^2 = S^2(\sqrt{3})/Z_2 \rightarrow S^4(1)$ , 称之为 *Veronese* 曲面, 它是极小的。

### 3.4 子流形变分公式

本节主要讨论子流形的变分公式。沿用前面的符号。主要思想取自于文献[4]。

设  $x: R(M, ds^2) \rightarrow (N, d\bar{s}^2)$  是子流形,  $X: (M, ds^2) \times (-\epsilon, \epsilon) \rightarrow (N, d\bar{s}^2)$  是其变分。

定义:

$$x_t: R = X(\cdot, t): RM \times \{t\} \rightarrow N, \quad t \in (-\epsilon, \epsilon).$$

那么每个  $x_t$  都是等距浸入, 而且  $x_0 = x$ 。

设  $d, d_M, d_{M \times (-\epsilon, \epsilon)} = d_M + dt \wedge \frac{\partial}{\partial t}$  是  $N, M, M \times (-\epsilon, \epsilon)$  上的微分算子, 设变分向量场为  $V = \sum_A V^A e_A$ , 即  $\frac{\partial X}{\partial t} = V$ 。通过拉回映射有:

$$X^* \sigma = \theta + dtV, \quad X^* \sigma^A = \theta^A + dtV^A, \quad (3-102)$$

$$X^* \sigma^i = \theta^i + dtV^i, \quad X^* \sigma^\alpha = dtV^\alpha, \quad (3-103)$$

$$X^* \omega = \phi + dtL, \quad X^* \omega_A^B = \phi_A^B + dtL_A^B, \quad (3-104)$$

$$X^* \omega_i^j = \phi_i^j + dtL_i^j, \quad X^* \omega_i^\alpha = \phi_i^\alpha + dtL_i^\alpha, \quad X^* \omega_\alpha^\beta = \phi_\alpha^\beta + dtL_\alpha^\beta, \quad (3-105)$$

$$X^* \Omega = \Phi + dt \wedge P, \quad X^* \Omega_A^B = \Phi_A^B + dt \wedge P_A^B, \quad (3-106)$$

$$X^* \Omega_i^j = \Phi_i^j + dt \wedge P_i^j, \quad X^* \Omega_i^\alpha = \Phi_i^\alpha + dt \wedge P_i^\alpha, \quad (3-107)$$

$$X^* \Omega_\alpha^\beta = \Phi_\alpha^\beta + dt \wedge P_\alpha^\beta. \quad (3-108)$$

其中

$$X^* \omega_A^B = \phi_A^B + dtL_A^B = \bar{\Gamma}_{Ai}^B \theta^i + dt \sum_C \bar{\Gamma}_{AC}^B V^C, \quad (3-109)$$

$$\phi_A^B = \bar{\Gamma}_{Ai}^B \theta^i, \quad L_A^B = \sum_C \bar{\Gamma}_{AC}^B V^C, \quad (3-110)$$

$$X^* \Omega_A^B = \frac{1}{2} \bar{R}_{ABCD} (\theta^C + dtV^C) \wedge (\theta^D + dtV^D), \quad (3-111)$$

$$= \frac{1}{2} \bar{R}_{ABCD} (\theta^C \wedge \theta^D + dt \wedge (V^C \theta^D - \theta^C V^D)), \quad (3-112)$$

$$= \frac{1}{2} \bar{R}_{ABij} \theta^i \wedge \theta^j + dt \wedge (\bar{R}_{ABCi} V^C \theta^i), \quad (3-113)$$

$$\Phi_A^B = \frac{1}{2} \bar{R}_{ABij} \theta^i \wedge \theta^j, \quad P_A^B = \bar{R}_{ABCi} V^C \theta^i. \quad (3-114)$$

定义 3.1: 定义张量

$$\bar{Z}_{ABi} = \bar{R}_{ABCi} V^C, \quad P_{AB} = \bar{Z}_{ABi} \theta^i.$$

对于以下三个方程, 通过拉回运算, 可以得到变分公式

$$\omega + \omega' = 0, \quad d\sigma - \sigma \wedge \omega = 0, \quad (3-115)$$

$$\Omega + \Omega' = 0, \quad d\omega - \omega \wedge \omega = \Omega, \quad (3-116)$$

$$\sigma \wedge \Omega = 0, \quad \sigma \wedge \Omega' = 0, \quad d\Omega = \omega \wedge \Omega - \Omega \wedge \omega. \quad (3-117)$$

拉回(3-115)

$$\phi + \phi' + dt(L + L') = 0, \quad (3-118)$$

$$(d_M + dt \wedge \frac{\partial}{\partial t})(\theta + dtV) - (\theta + dtV) \wedge (\phi + dtL) = 0, \quad (3-119)$$

$$\phi + \phi' = 0, \quad L + L' = 0, \quad (3-120)$$

$$d_M \theta - \theta \wedge \phi + dt \wedge (\frac{\partial \theta}{\partial t} - d_M V - V\phi + \theta L) = 0, \quad (3-121)$$

$$d_M \theta - \theta \wedge \phi = 0, \quad \frac{\partial \theta}{\partial t} = d_M V + V\phi - \theta L, \quad (3-122)$$

$$\frac{\partial \theta^j}{\partial t} = d_M V^j + V^j \phi'_I + V^A \phi'_A - \theta^j L'_I - \theta^A L'_A \quad (3-123)$$

$$= DV^j + V^A \phi'_A - \theta^j L'_I, \quad (3-124)$$

$$\frac{\partial \theta^i}{\partial t} = \sum_j (V^j - \sum_\alpha h_{ij}^\alpha V^\alpha - L_j^i) \theta^j, \quad (3-125)$$

$$\frac{\partial \theta^A}{\partial t} = d_M V^A + V^A \phi'_A + V^j \phi'_j - \theta^j L'_I - \theta^A L'_A \quad (3-126)$$

$$= DV^A + V^j \phi'_j - \theta^j L'_I, \quad (3-127)$$

$$L_i^\alpha = V_{,i}^\alpha + \sum_j h_{ij}^\alpha V^j, \quad (3-128)$$

$$L_{i,j}^\alpha = V_{,ij}^\alpha + \sum_p h_{ip}^\alpha V_p^j + \sum_p h_{ij,p}^\alpha V^p + \sum_p \bar{R}_{ijp}^\alpha V^p. \quad (3-129)$$

拉回(3-116)

$$\Phi + \Phi' + dt(P + P') = 0, \quad \Phi + \Phi' = 0, \quad P + P' = 0, \quad (3-130)$$

$$\Phi + dt \wedge P = (d_M + dt \wedge \frac{\partial}{\partial t})(\phi + dtL) - (\phi + dtL) \wedge (\phi + dtL), \quad (3-131)$$

$$\Phi = d_M \phi - \phi \wedge \phi, \quad \frac{\partial \phi}{\partial t} = d_M L + L\phi - \phi L + P. \quad (3-132)$$

对于矩阵  $\Phi = d_M \phi - \phi \wedge \phi$  的第一部分,  $L_i^j$  不是张量, 但是可以形式地记为



$$\frac{\partial \theta_l^I}{\partial t} = d_M L_l^I + L_l^I \phi_l^I + L_l^A \phi_A^I - \phi_l^I L_l^I - \phi_l^A L_A^I + P_l^I \quad (3-133)$$

$$= DL_l^I + L_l^A \phi_A^I - \phi_l^A L_A^I + P_l^I, \quad (3-134)$$

$$\begin{aligned} \frac{\partial \Gamma_{ik}^j}{\partial t} &= L_{i,k}^j + \sum_{\alpha} h_{ik}^{\alpha} L_j^{\alpha} - \sum_{\alpha} L_i^{\alpha} h_{jk}^{\alpha} + \bar{Z}_{ijk} \\ &\quad - \sum_p \Gamma_{ip}^j V_{,k}^p + \sum_{p\alpha} \Gamma_{ip}^j h_{pk}^{\alpha} V^{\alpha} + \sum_p \Gamma_{ip}^j L_k^p. \end{aligned} \quad (3-135)$$

对于矩阵  $\Phi = d_M \phi - \phi \wedge \phi$  的第二部分,  $L_i^{\alpha}$  是张量, 记为

$$\frac{\partial \theta_l^A}{\partial t} = d_M L_l^A + L_l^I \phi_l^A + L_l^I \phi_A^A - \phi_l^I L_l^A - \phi_l^A L_A^A + P_l^A \quad (3-136)$$

$$= DL_l^A + L_l^I \phi_l^A - \phi_l^A L_A^A + P_l^A, \quad (3-137)$$

$$\begin{aligned} \frac{\partial h_{ij}^{\alpha}}{\partial t} &= L_{i,j}^{\alpha} + \sum_p L_i^p h_{pj}^{\alpha} - \sum_{\beta} h_{ij}^{\beta} L_{\beta}^{\alpha} + \bar{Z}_{ij}^{\alpha} \\ &\quad - \sum_p h_{ip}^{\alpha} V_{,j}^p + \sum_{p\beta} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} + \sum_p h_{ip}^{\alpha} L_j^p \end{aligned} \quad (3-138)$$

$$\begin{aligned} &= V_{,ij}^{\alpha} + \sum_p h_{ij,p}^{\alpha} V^p + \sum_p h_{pj}^{\alpha} L_i^p + \sum_p h_{ip}^{\alpha} L_j^p - \sum_{\beta} h_{ij}^{\beta} L_{\beta}^{\alpha} \\ &\quad + \sum_{p\beta} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} - \sum_{\beta} \bar{R}_{ij\beta}^{\alpha} V^{\beta}. \end{aligned} \quad (3-139)$$

对于矩阵  $\Phi = d_M \phi - \phi \wedge \phi$  的第四部分,  $L_{\alpha}^{\beta}$  不是张量, 但是可以形式地记为

$$\frac{\partial \theta_A^A}{\partial t} = d_M L_A^A + L_A^I \phi_A^A + L_A^I \phi_l^A - \phi_A^I L_A^A - \phi_l^A L_l^A + P_A^A \quad (3-140)$$

$$= DL_A^A + L_A^I \phi_l^A - \phi_l^A L_l^A + P_A^A, \quad (3-141)$$

$$\begin{aligned} \frac{\partial \Gamma_{\alpha i}^{\beta}}{\partial t} &= L_{\alpha,i}^{\beta} + \sum_p L_p^{\beta} h_{pi}^{\alpha} - \sum_p L_p^{\alpha} h_{pi}^{\beta} + \bar{Z}_{\alpha\beta i} \\ &\quad - \sum_p \Gamma_{\alpha p}^{\beta} V_{,i}^p + \sum_p \Gamma_{\alpha p}^{\beta} h_{p\alpha}^{\gamma} V^{\gamma} + \sum_p \Gamma_{\alpha p}^{\beta} L_i^p. \end{aligned} \quad (3-142)$$

拉回(3-117)

$$(\theta + dtV) \wedge (\Phi + dtP) = 0, \quad \theta \wedge \Phi = 0, \quad V\Phi - \theta \wedge P = 0. \quad (3-143)$$

上式是 Bianchi 恒等式, 对于(3-117)的后半部分, 有

$$LHS = (d_M + dt \wedge \frac{\partial}{\partial t})(\Phi + dtP) = d_M \Phi + dt \wedge (\frac{\partial \Phi}{\partial t} - d_M P), \quad (3-144)$$

$$RHS = (\phi + dtL) \wedge (\Phi + dtP) - (\Phi + dtP) \wedge (\phi + dtL) \quad (3-145)$$

$$= \phi \wedge \Phi - \Phi \wedge \phi + dt(L\Phi - \phi P - P\phi - \Phi L), \quad (3-146)$$

$$d_M \Phi = \phi \wedge \Phi - \Phi \wedge \phi, \quad \frac{\partial \Phi}{\partial t} = d_M P + L\Phi - \phi P - P\phi - \Phi L \quad (3-147)$$

对于矩阵(3-56)的第一部分

$$\begin{aligned}\frac{\partial \Phi_l'}{\partial t} &= d_M P_l' - \phi_l' P_l' - P_l' \phi_l' - \phi_l' P_A' - P_A' \phi_l' + L_l' \Phi_l' + L_A' \Phi_A' - \Phi_l' L_l' - \Phi_A' L_A', \\ \frac{\partial \Phi_{ij}}{\partial t} &= \bar{Z}_{ijl,k} \theta^k \wedge \theta^l + \sum_{\alpha} h_{ik}^{\alpha} \bar{Z}_{jl}^{\alpha} \theta^k \wedge \theta^l + \sum_{\alpha} \bar{Z}_{ik}^{\alpha} h_{jl}^{\alpha} \theta^k \wedge \theta^l \\ &\quad + \sum_p L_{ip} \Phi_{pj} - \sum_p \Phi_{ip} L_{pj} + \sum_{\alpha} \Phi_i^{\alpha} L_j^{\alpha} - \sum_{\alpha} L_i^{\alpha} \Phi_j^{\alpha},\end{aligned}\quad (3-148)$$

$$\begin{aligned}\frac{\partial \bar{R}_{ijkl}}{\partial t} &= (\bar{Z}_{ijl,k} - \bar{Z}_{ijk,l}) + \sum_{\alpha} (h_{ik}^{\alpha} \bar{Z}_{jl}^{\alpha} - h_{il}^{\alpha} \bar{Z}_{jk}^{\alpha} + \bar{Z}_{ik}^{\alpha} h_{jl}^{\alpha} - \bar{Z}_{il}^{\alpha} h_{jk}^{\alpha}) \\ &\quad + \sum_A (\bar{R}_{ikl}^A L_j^A - L_i^A \bar{R}_{jkl}^A) - \sum_p (\bar{R}_{ijpl} V_{p,k}^p + \bar{R}_{ijkp} V_{p,l}^p) \\ &\quad + \sum_{p\alpha} (\bar{R}_{ijpl} h_{pk}^{\alpha} V^{\alpha} + \bar{R}_{ijkp} h_{pl}^{\alpha} V^{\alpha}) + \sum_p (\bar{R}_{ijpl} L_k^p + \bar{R}_{ijkp} L_l^p).\end{aligned}\quad (3-149)$$

对于矩阵(3-56)的第二部分

$$\frac{\partial \Phi_l^A}{\partial t} = d_M P_l^A - \phi_l^A P_l^A - P_l^A \phi_l^A - \phi_l^A P_A^A - P_A^A \phi_l^A + L_l^A \Phi_l^A + L_A^A \Phi_A^A - \Phi_l^A L_l^A - \Phi_A^A L_A^A, \quad (3-150)$$

$$\frac{\partial \Phi_i^{\alpha}}{\partial t} = D P_i^{\alpha} - \sum_p P_{ip} \phi_p^{\alpha} - \sum_{\beta} \phi_i^{\beta} P_{\beta}^{\alpha} + \sum_A (L_i^A \Phi_A^{\alpha} - \Phi_i^A L_A^{\alpha}), \quad (3-151)$$

$$\begin{aligned}\frac{\partial \bar{R}_{ijk}^{\alpha}}{\partial t} &= (\bar{Z}_{ik,j}^{\alpha} - \bar{Z}_{ij,k}^{\alpha}) + \sum_p (\bar{Z}_{ipk} h_{pj}^{\alpha} - \bar{Z}_{ipj} h_{pk}^{\alpha}) + \sum_{\beta} (h_{ik}^{\beta} \bar{Z}_{\beta j}^{\alpha} - h_{ij}^{\beta} \bar{Z}_{\beta k}^{\alpha}) \\ &\quad + \sum_A (L_i^A \bar{R}_{Ajk}^{\alpha} - \bar{R}_{ijk}^{\alpha} L_A^A) - \sum_p (\bar{R}_{ipk}^{\alpha} V_j^p + \bar{R}_{ijp}^{\alpha} V_{p,k}^p) + \sum_{p\beta} (\bar{R}_{ipk}^{\alpha} h_{pj}^{\beta} V^{\beta} + \bar{R}_{ijp}^{\alpha} h_{pk}^{\beta} V^{\beta}) \\ &\quad + \sum_p (\bar{R}_{ipk}^{\alpha} L_j^p + \bar{R}_{ijp}^{\alpha} L_k^p).\end{aligned}\quad (3-152)$$

对于矩阵(3-56)的第四部分

$$\frac{\partial \Phi_A^A}{\partial t} = d_M P_A^A - \phi_A^A P_A^A - P_A^A \phi_A^A - \phi_A^A P_l^A - \phi_l^A P_A^A + L_A^A \Phi_A^A + L_l^A \Phi_l^A - \Phi_A^A L_l^A - \Phi_l^A L_A^A, \quad (3-153)$$

$$\frac{\partial \Phi_{\alpha}^{\beta}}{\partial t} = D P_{\alpha}^{\beta} + \sum_p P_{p\alpha}^{\beta} \phi_p^{\beta} + \sum_p \phi_p^{\alpha} P_p^{\beta} + \sum_A (L_{\alpha}^A \Phi_A^{\beta} - \Phi_{\alpha}^A L_A^{\beta}), \quad (3-154)$$

$$\begin{aligned}\frac{\partial \bar{R}_{\alpha\beta ij}}{\partial t} &= (\bar{Z}_{\alpha\beta i,j} - \bar{Z}_{\alpha\beta j,i}) + \sum_p (\bar{Z}_{pi}^{\alpha} h_{pj}^{\beta} - \bar{Z}_{pj}^{\alpha} h_{pi}^{\beta} + h_{ip}^{\alpha} \bar{Z}_{pj}^{\beta} - h_{jp}^{\alpha} \bar{Z}_{pi}^{\beta}) \\ &\quad + \sum_A (\bar{R}_{Aij}^{\alpha} L_A^{\beta} - L_A^{\alpha} \bar{R}_{Aij}^{\beta}) - \sum_p (\bar{R}_{\alpha\beta pj} V_{p,i}^p + \bar{R}_{\alpha\beta ip} V_{p,j}^p) \\ &\quad + \sum_{p\gamma} (\bar{R}_{\alpha\beta pj} h_{ip}^{\gamma} V^{\gamma} + \bar{R}_{\alpha\beta ip} h_{jp}^{\gamma} V^{\gamma}) + \sum_p (\bar{R}_{\alpha\beta pj} L_i^p + \bar{R}_{\alpha\beta ip} L_j^p).\end{aligned}\quad (3-155)$$

综上,证明了以下变分基本公式。

**定理 3.3:** 设  $x: M \rightarrow N$  是子流形,  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 令

$\bar{Z}_{ABi} = \sum_C \bar{R}_{ABCi} V^C$ , 张量的变分公式为

$$\frac{\partial \theta^j}{\partial t} = \sum_j (V_j^j - \sum_{\alpha} h_{ij}^{\alpha} V^{\alpha} - L_j^i) \theta^j, \quad \frac{\partial dv}{\partial t} = (\operatorname{div} V^{\top} - n \sum_{\alpha} H^{\alpha} V^{\alpha}) dv, \quad (3-156)$$

$$\frac{\partial \Gamma_{ik}^j}{\partial t} = L_{i,k}^j + \sum_{\alpha} h_{ik}^{\alpha} L_j^{\alpha} - \sum_{\alpha} L_i^{\alpha} h_{jk}^{\alpha} + \bar{Z}_{ijk} - \sum_p \Gamma_{ip}^j V_{,k}^p + \sum_{p\alpha} \Gamma_{ip}^j h_{pk}^{\alpha} V^{\alpha} + \sum_p \Gamma_{ip}^j L_k^p, \quad (3-157)$$

$$\begin{aligned} \frac{\partial h_{ij}^{\alpha}}{\partial t} &= V_{,ij}^{\alpha} + \sum_p h_{ij,p}^{\alpha} V^p + \sum_p h_{pj}^{\alpha} L_i^p + \sum_p h_{ip}^{\alpha} L_j^p - \sum_{\beta} h_{ij}^{\beta} L_{\beta}^{\alpha} \\ &\quad + \sum_{p\beta} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} - \sum_{\beta} \bar{R}_{ij\beta}^{\alpha} V^{\beta}, \end{aligned} \quad (3-158)$$

$$\begin{aligned} \frac{\partial \Gamma_{\alpha i}^{\beta}}{\partial t} &= L_{\alpha,i}^{\beta} + \sum_p L_p^{\beta} h_{pi}^{\alpha} - \sum_p L_p^{\alpha} h_{pi}^{\beta} + \bar{Z}_{\alpha\beta i} - \sum_p \Gamma_{\alpha p}^{\beta} V_{,i}^p + \sum_p \Gamma_{\alpha p}^{\beta} h_{pi}^{\gamma} V^{\gamma} \\ &\quad + \sum_p \Gamma_{\alpha p}^{\beta} L_i^p, \end{aligned} \quad (3-159)$$

$$\begin{aligned} \frac{\partial \bar{R}_{ijkl}}{\partial t} &= (\bar{Z}_{ijl,k} - \bar{Z}_{ijk,l}) + \sum_{\alpha} (h_{ik}^{\alpha} \bar{Z}_{jl}^{\alpha} - h_{il}^{\alpha} \bar{Z}_{jk}^{\alpha} + \bar{Z}_{ik}^{\alpha} h_{jl}^{\alpha} - \bar{Z}_{il}^{\alpha} h_{jk}^{\alpha}) \\ &\quad + \sum_A (\bar{R}_{ikl}^A L_j^A - L_i^A \bar{R}_{jkl}^A) - \sum_p (\bar{R}_{ijpl} V_{,k}^p + \bar{R}_{ijkp} V_{,l}^p) \\ &\quad + \sum_{p\alpha} (\bar{R}_{ijpl} h_{pk}^{\alpha} V^{\alpha} + \bar{R}_{ijkp} h_{pl}^{\alpha} V^{\alpha}) + \sum_p (\bar{R}_{ijpl} L_k^p + \bar{R}_{ijkp} L_l^p), \end{aligned} \quad (3-160)$$

$$\begin{aligned} \frac{\partial \bar{R}_{ijk}^{\alpha}}{\partial t} &= (\bar{Z}_{ik,j}^{\alpha} - \bar{Z}_{ij,k}^{\alpha}) + \sum_p (\bar{Z}_{ipk} h_{pj}^{\alpha} - \bar{Z}_{ijp} h_{pk}^{\alpha}) + \sum_{\beta} (h_{ik}^{\beta} \bar{Z}_{\beta j}^{\alpha} - h_{ij}^{\beta} \bar{Z}_{\beta k}^{\alpha}) \\ &\quad + \sum_A (L_i^A \bar{R}_{Aj k}^{\alpha} - \bar{R}_{ijk}^A L_A^{\alpha}) - \sum_p (\bar{R}_{ipk}^{\alpha} V_{,j}^p + \bar{R}_{ijp}^{\alpha} V_{,k}^p) \\ &\quad + \sum_{p\beta} (\bar{R}_{ipk}^{\alpha} h_{pj}^{\beta} V^{\beta} + \bar{R}_{ijp}^{\alpha} h_{pk}^{\beta} V^{\beta}) + \sum_p (\bar{R}_{ipk}^{\alpha} L_j^p + \bar{R}_{ijp}^{\alpha} L_k^p), \end{aligned} \quad (3-161)$$

$$\begin{aligned} \frac{\partial \bar{R}_{\alpha\beta ij}}{\partial t} &= (\bar{Z}_{\alpha\beta j,i} - \bar{Z}_{\alpha\beta i,j}) + \sum_p (\bar{Z}_{pi} h_{pj}^{\beta} - \bar{Z}_{pj} h_{pi}^{\beta} + h_{ip}^{\alpha} \bar{Z}_{pj}^{\beta} - h_{jp}^{\alpha} \bar{Z}_{pi}^{\beta}) \\ &\quad + \sum_A (\bar{R}_{Aij}^{\beta} L_A^{\beta} - L_A^{\alpha} \bar{R}_{Aij}^{\beta}) - \sum_p (\bar{R}_{\alpha\beta pj} V_{,i}^p + \bar{R}_{\alpha\beta ip} V_{,j}^p) \\ &\quad + \sum_{p\gamma} (\bar{R}_{\alpha\beta pj} h_{ip}^{\gamma} V^{\gamma} + \bar{R}_{\alpha\beta ip} h_{jp}^{\gamma} V^{\gamma}) + \sum_p (\bar{R}_{\alpha\beta pj} L_i^p + \bar{R}_{\alpha\beta ip} L_j^p). \end{aligned} \quad (3-162)$$

**注释 3.3:** 关于余标架、体积与第二基本型的变分公式可参见文献[4], 其余的公式都是新推导的。

特别的, 作如下记号,

$$\bar{R}_{AB} = \sum_C \bar{R}_{ACCB}, \quad \bar{R}_{AB}^{\top} = \sum_i \bar{R}_{AiiB}, \quad \bar{R}_{AB}^{\perp} = \sum_{\alpha} \bar{R}_{A\alpha\alpha B}.$$

分别称为流形  $N$  的 Ricci 曲率, 切 Ricci 曲率, 法 Ricci 曲率。

观察上面的定理, 我们发现, 黎曼张量  $\bar{R}_{i\alpha jk}$ ,  $\bar{R}_{\alpha ijk}$ ,  $\bar{R}_{\alpha\beta ij}$  的变分公式我们已经获得, 但是其他类型的黎曼张量比如  $\bar{R}_{ijk\alpha}$  的变分公式并没有获得, 实际上我们可以通过更加一般的方式获得。首先定义流形  $N$  上的黎曼张量  $\bar{R}_{ABCD}$  的协变导数为

$$\bar{R}_{ABCD;E} \sigma^E = d \bar{R}_{ABCD} - \bar{R}_{FBCD} \omega_A^F - \bar{R}_{AFCD} \omega_B^F - \bar{R}_{ABFD} \omega_C^F - \bar{R}_{ABCF} \omega_D^F.$$

通过拉回映射, 我们知道

$$x^*(\bar{R}_{ABCD;E}\sigma^E) = (T1)x^*(d\bar{R}_{ABCD}) - (T2)x^*(\bar{R}_{FBCD}\omega_A^F) \\ - (T3)x^*(\bar{R}_{AFCD}\omega_B^F) - (T4)x^*(\bar{R}_{ABFD}\omega_C^F) - (T5)x^*(\bar{R}_{ABCF}\omega_D^F), \quad (3-163)$$

$$RHS = x^*(\bar{R}_{ABCD;E}\sigma^E) = \sum_i \bar{R}_{ABCD;i}\theta^i + dt \wedge \left( \sum_E \bar{R}_{ABCD;E}V^E \right), \quad (3-164)$$

$$T1 = x^*(d\bar{R}_{ABCD}) = d_M \bar{R}_{ABCD} + dt \wedge \frac{\partial}{\partial t} \bar{R}_{ABCD}, \quad (3-165)$$

$$T2 = x^*(\bar{R}_{FBCD}\omega_A^F) = \sum_F \bar{R}_{FBCD}(\phi_A^F + dtL_A^F), \quad (3-166)$$

$$T3 = x^*(\bar{R}_{AFCD}\omega_B^F) = \sum_F \bar{R}_{AFCD}(\phi_B^F + dtL_B^F), \quad (3-167)$$

$$T4 = x^*(\bar{R}_{ABFD}\omega_C^F) = \sum_F \bar{R}_{ABFD}(\phi_C^F + dtL_C^F), \quad (3-168)$$

$$T5 = x^*(\bar{R}_{ABCF}\omega_D^F) = \sum_F \bar{R}_{ABCF}(\phi_D^F + dtL_D^F), \quad (3-169)$$

$$LHS = d_M \bar{R}_{ABCD} - \sum_F \bar{R}_{FBCD}\phi_A^F - \sum_F \bar{R}_{AFCD}\phi_B^F - \sum_F \bar{R}_{ABFD}\phi_C^F - \sum_F \bar{R}_{ABCF}\phi_D^F, \\ + dt \wedge \left( \frac{\partial}{\partial t} \bar{R}_{ABCD} - \sum_F \bar{R}_{FBCD}L_A^F - \sum_F \bar{R}_{AFCD}L_B^F - \sum_F \bar{R}_{ABFD}L_C^F \right. \\ \left. - \sum_F \bar{R}_{ABCF}L_D^F \right), \quad (3-170)$$

$$RHS = LHS, \quad (3-171)$$

$$\sum_i \bar{R}_{ABCD;i}\theta^i = d_M \bar{R}_{ABCD} - \sum_F \bar{R}_{FBCD}\phi_A^F - \sum_F \bar{R}_{AFCD}\phi_B^F \\ - \sum_F \bar{R}_{ABFD}\phi_C^F - \sum_F \bar{R}_{ABCF}\phi_D^F, \quad (3-172)$$

$$\frac{\partial}{\partial t} \bar{R}_{ABCD} = \sum_E \bar{R}_{ABCD;E}V^E + \sum_F \bar{R}_{FBCD}L_A^F + \sum_F \bar{R}_{AFCD}L_B^F \\ + \sum_F \bar{R}_{ABFD}L_C^F + \sum_F \bar{R}_{ABCF}L_D^F. \quad (3-173)$$

因此,我们可以总结以上的变分公式为以下定理.

**定理 3.4:** 设  $x:M \rightarrow N$  是子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 我们有

$$\sum_i \bar{R}_{ABCD;i}\theta^i = d_M \bar{R}_{ABCD} - \sum_F \bar{R}_{FBCD}\phi_A^F - \sum_F \bar{R}_{AFCD}\phi_B^F \\ - \sum_F \bar{R}_{ABFD}\phi_C^F - \sum_F \bar{R}_{ABCF}\phi_D^F, \quad (3-174)$$

$$\frac{\partial}{\partial t} \bar{R}_{ABCD} = \sum_E \bar{R}_{ABCD;E}V^E + \sum_F \bar{R}_{FBCD}L_A^F + \sum_F \bar{R}_{AFCD}L_B^F \\ + \sum_F \bar{R}_{ABFD}L_C^F + \sum_F \bar{R}_{ABCF}L_D^F. \quad (3-175)$$

从定理 3.4 的第一个公式出发, 可以得到张量  $\bar{R}_{ABCD}$  在流形  $N$  上的协变导数  $\bar{R}_{ABCD;i}$  与其在流形  $M$  的拉回从  $x^*TN$  上的协变导数  $\bar{R}_{ABCD,i}$  之间的差异, 对于不同的指标集合  $ABCD$ , 差异公式也不相同, 我们作如下推导.

$$\begin{aligned}
\bar{R}_{ijkl;p}\theta^p &= d_M \bar{R}_{ijkl} - \sum_A \bar{R}_{Ajk l} \phi_i^A - \sum_A \bar{R}_{iAkl} \phi_j^A - \sum_A \bar{R}_{ijAl} \phi_k^A - \sum_A \bar{R}_{ijkA} \phi_l^A \\
&= d_M \bar{R}_{ijkl} - \sum_q \bar{R}_{qjkl} \phi_i^q - \sum_q \bar{R}_{iqkl} \phi_j^q - \sum_q \bar{R}_{ijql} \phi_k^q - \sum_q \bar{R}_{ijkq} \phi_l^q \\
&\quad - \sum_\alpha \bar{R}_{\alpha jkl} \phi_i^\alpha - \sum_\alpha \bar{R}_{i\alpha kl} \phi_j^\alpha - \sum_\alpha \bar{R}_{ij\alpha l} \phi_k^\alpha - \sum_\alpha \bar{R}_{ijk\alpha} \phi_l^\alpha \\
&= \bar{R}_{ijkl, p} \theta^p - \sum_\alpha \bar{R}_{\alpha jkl} h_{ip}^\alpha \theta^p - \sum_\alpha \bar{R}_{i\alpha kl} h_{jp}^\alpha \theta^p \\
&\quad - \sum_\alpha \bar{R}_{ij\alpha l} h_{kp}^\alpha \theta^p - \sum_\alpha \bar{R}_{ijk\alpha} h_{lp}^\alpha \theta^p, \tag{3-176}
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{ijk\alpha;p}\theta^p &= d_M \bar{R}_{ijk\alpha} - \sum_A \bar{R}_{Ajk\alpha} \phi_i^A - \sum_A \bar{R}_{iA\alpha k} \phi_j^A - \sum_A \bar{R}_{ijA\alpha} \phi_k^A - \sum_A \bar{R}_{ijkA} \phi_\alpha^A \\
&= d_M \bar{R}_{ijk\alpha} - \sum_q \bar{R}_{qjk\alpha} \phi_i^q - \sum_q \bar{R}_{iq\alpha k} \phi_j^q - \sum_q \bar{R}_{ijq\alpha} \phi_k^q - \sum_\beta \bar{R}_{ijk\beta} \phi_\alpha^\beta \\
&\quad - \sum_\beta \bar{R}_{\beta jk\alpha} \phi_i^\beta - \sum_\beta \bar{R}_{i\beta\alpha k} \phi_j^\beta - \sum_\beta \bar{R}_{ij\beta\alpha} \phi_k^\beta - \sum_q \bar{R}_{ijkq} \phi_\alpha^q \\
&= \bar{R}_{ijk\alpha, p} \theta^p - \sum_\beta \bar{R}_{\beta jk\alpha} h_{ip}^\beta \theta^p - \sum_\beta \bar{R}_{i\beta\alpha k} h_{jp}^\beta \theta^p \\
&\quad - \sum_\beta \bar{R}_{ij\beta\alpha} h_{kp}^\beta \theta^p + \sum_q \bar{R}_{ijkq} h_{\alpha p}^q \theta^p, \tag{3-177}
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{ij\alpha\beta;p}\theta^p &= d_M \bar{R}_{ij\alpha\beta} - \sum_A \bar{R}_{A j\alpha\beta} \phi_i^A - \sum_A \bar{R}_{iA\alpha\beta} \phi_j^A - \sum_A \bar{R}_{ijA\alpha} \phi_\beta^A - \sum_A \bar{R}_{ij\alpha A} \phi_\beta^A \\
&= d_M \bar{R}_{ij\alpha\beta} - \sum_q \bar{R}_{q j\alpha\beta} \phi_i^q - \sum_q \bar{R}_{i q\alpha\beta} \phi_j^q - \sum_\gamma \bar{R}_{ij\gamma\alpha} \phi_\beta^\gamma - \sum_\gamma \bar{R}_{ij\alpha\gamma} \phi_\beta^\gamma \\
&\quad - \sum_\gamma \bar{R}_{\gamma j\alpha\beta} \phi_i^\gamma - \sum_\gamma \bar{R}_{i\gamma\alpha\beta} \phi_j^\gamma + \sum_q \bar{R}_{ijq\beta} \phi_\alpha^q - \sum_q \bar{R}_{ij\alpha q} \phi_\beta^q \\
&= \bar{R}_{ij\alpha\beta, p} \theta^p - \sum_\gamma \bar{R}_{\gamma j\alpha\beta} h_{ip}^\gamma \theta^p - \sum_\gamma \bar{R}_{i\gamma\alpha\beta} h_{jp}^\gamma \theta^p \\
&\quad + \sum_q \bar{R}_{ijq\beta} h_{\alpha p}^q \theta^p + \sum_q \bar{R}_{ij\alpha q} h_{\beta p}^q \theta^p, \tag{3-178}
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{i\alpha\beta\gamma;p}\theta^p &= d_M \bar{R}_{i\alpha\beta\gamma} - \sum_A \bar{R}_{A\alpha\beta\gamma} \phi_i^A - \sum_A \bar{R}_{iA\beta\gamma} \phi_\alpha^A - \sum_A \bar{R}_{i\alpha A\beta} \phi_\gamma^A - \sum_A \bar{R}_{i\alpha\beta A} \phi_\gamma^A \\
&= d_M \bar{R}_{i\alpha\beta\gamma} - \sum_q \bar{R}_{q\alpha\beta\gamma} \phi_i^q - \sum_\gamma \bar{R}_{i\gamma\beta\gamma} \phi_\alpha^\gamma - \sum_q \bar{R}_{i\alpha q\beta} \phi_\gamma^q - \sum_\gamma \bar{R}_{i\alpha\gamma\beta} \phi_\gamma^\gamma \\
&\quad - \sum_\gamma \bar{R}_{\gamma\alpha\beta\gamma} \phi_i^\gamma + \sum_q \bar{R}_{i\alpha q\beta} \phi_\gamma^q - \sum_\gamma \bar{R}_{i\alpha\gamma\beta} \phi_\gamma^\gamma + \sum_q \bar{R}_{i\alpha\beta q} \phi_\gamma^q \\
&= \bar{R}_{i\alpha\beta\gamma, p} \theta^p - \sum_\gamma \bar{R}_{\gamma\alpha\beta\gamma} h_{ip}^\gamma \theta^p + \sum_q \bar{R}_{i\alpha q\beta} h_{\gamma p}^q \theta^p \\
&\quad - \sum_\gamma \bar{R}_{i\alpha\gamma\beta} h_{jp}^\gamma \theta^p + \sum_q \bar{R}_{i\alpha\beta q} h_{\gamma p}^q \theta^p, \tag{3-179}
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{i\alpha\beta\gamma;p}\theta^p &= d_M \bar{R}_{i\alpha\beta\gamma} - \sum_A \bar{R}_{A\alpha\beta\gamma} \phi_i^A - \sum_A \bar{R}_{iA\beta\gamma} \phi_\alpha^A - \sum_A \bar{R}_{i\alpha A\gamma} \phi_\beta^A - \sum_A \bar{R}_{i\alpha\beta A} \phi_\gamma^A \\
&= d_M \bar{R}_{i\alpha\beta\gamma} - \sum_q \bar{R}_{q\alpha\beta\gamma} \phi_i^q - \sum_\delta \bar{R}_{i\delta\beta\gamma} \phi_\alpha^\delta - \sum_\delta \bar{R}_{i\alpha\delta\gamma} \phi_\beta^\delta - \sum_\delta \bar{R}_{i\alpha\beta\delta} \phi_\gamma^\delta \\
&\quad - \sum_\delta \bar{R}_{\delta\alpha\beta\gamma} \phi_i^\delta - \sum_q \bar{R}_{i\alpha\beta q} \phi_\gamma^q - \sum_q \bar{R}_{i\alpha q\gamma} \phi_\beta^q - \sum_q \bar{R}_{i\alpha\beta q} \phi_\gamma^q
\end{aligned}$$

$$\begin{aligned}
&= \bar{R}_{\alpha\beta\gamma, p} \theta^p - \sum_{\delta} \bar{R}_{\delta\alpha\beta\gamma} h_{ip}^{\delta} \theta^p + \sum_q \bar{R}_{i\alpha\beta\gamma} h_{qp}^{\alpha} \theta^p \\
&\quad + \sum_q \bar{R}_{i\alpha\beta\gamma} h_{qp}^{\beta} \theta^p + \sum_q \bar{R}_{i\alpha\beta\gamma} h_{qp}^{\gamma} \theta^p, \quad (3-180)
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{\alpha\beta\gamma\delta; p} \theta^p &= d_M \bar{R}_{\alpha\beta\gamma\delta} - \sum_{\lambda} \bar{R}_{\lambda\beta\gamma\delta} \phi_{\alpha}^{\lambda} - \sum_{\lambda} \bar{R}_{\alpha\lambda\gamma\delta} \phi_{\beta}^{\lambda} - \sum_{\lambda} \bar{R}_{\alpha\beta\lambda\delta} \phi_{\gamma}^{\lambda} - \sum_{\lambda} \bar{R}_{\alpha\beta\gamma\lambda} \phi_{\delta}^{\lambda} \\
&= d_M \bar{R}_{\alpha\beta\gamma\delta} - \sum_{\eta} \bar{R}_{\eta\beta\gamma\delta} \phi_{\alpha}^{\eta} - \sum_{\eta} \bar{R}_{\alpha\eta\gamma\delta} \phi_{\beta}^{\eta} - \sum_{\eta} \bar{R}_{\alpha\beta\eta\delta} \phi_{\gamma}^{\eta} - \sum_{\eta} \bar{R}_{\alpha\beta\gamma\eta} \phi_{\delta}^{\eta} \\
&\quad - \sum_q \bar{R}_{q\beta\gamma\delta} \phi_{\alpha}^q - \sum_q \bar{R}_{\alpha q\gamma\delta} \phi_{\beta}^q - \sum_q \bar{R}_{\alpha\beta q\delta} \phi_{\gamma}^q - \sum_q \bar{R}_{\alpha\beta\gamma q} \phi_{\delta}^q \\
&= \bar{R}_{\alpha\beta\gamma\delta, p} \theta^p + \sum_q \bar{R}_{q\beta\gamma\delta} h_{qp}^{\alpha} \theta^p + \sum_q \bar{R}_{\alpha q\gamma\delta} h_{qp}^{\beta} \theta^p \\
&\quad + \sum_q \bar{R}_{\alpha\beta q\delta} h_{qp}^{\gamma} \theta^p + \sum_q \bar{R}_{\alpha\beta\gamma q} h_{qp}^{\delta} \theta^p. \quad (3-181)
\end{aligned}$$

综上, 我们有协变导数的差异公式。

**定理 3.5:** 设  $x: M \rightarrow N$  是子流形, 协变导数的差异公式如下

$$\begin{aligned}
\bar{R}_{ijkl; p} &= \bar{R}_{ijkl, p} - \sum_{\alpha} \bar{R}_{\alpha jkl} h_{ip}^{\alpha} - \sum_{\alpha} \bar{R}_{i\alpha kl} h_{jp}^{\alpha} \\
&\quad - \sum_{\alpha} \bar{R}_{ij\alpha l} h_{kp}^{\alpha} - \sum_{\alpha} \bar{R}_{ijk\alpha} h_{lp}^{\alpha}, \quad (3-182)
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{ijk\alpha; p} &= \bar{R}_{ijk\alpha, p} - \sum_{\beta} \bar{R}_{\beta jk\alpha} h_{ip}^{\beta} - \sum_{\beta} \bar{R}_{i\beta k\alpha} h_{jp}^{\beta} \\
&\quad - \sum_{\beta} \bar{R}_{ij\beta\alpha} h_{kp}^{\beta} + \sum_q \bar{R}_{ijkq} h_{qp}^{\alpha}, \quad (3-183)
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{ij\alpha\beta; p} &= \bar{R}_{ij\alpha\beta, p} - \sum_{\gamma} \bar{R}_{ij\alpha\beta} h_{ip}^{\gamma} - \sum_{\gamma} \bar{R}_{i\gamma\alpha\beta} h_{jp}^{\gamma} \\
&\quad + \sum_q \bar{R}_{ijq\beta} h_{qp}^{\alpha} + \sum_q \bar{R}_{ij\alpha q} h_{qp}^{\beta}, \quad (3-184)
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{i\alpha\beta\gamma; p} &= \bar{R}_{i\alpha\beta\gamma, p} - \sum_{\gamma} \bar{R}_{\gamma\alpha\beta\gamma} h_{ip}^{\gamma} + \sum_q \bar{R}_{i\alpha\beta q} h_{qp}^{\gamma} \\
&\quad - \sum_{\gamma} \bar{R}_{i\alpha\gamma\beta} h_{jp}^{\gamma} + \sum_q \bar{R}_{i\alpha q\beta} h_{qp}^{\gamma}, \quad (3-185)
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{i\alpha\beta\gamma; p} &= \bar{R}_{i\alpha\beta\gamma, p} - \sum_{\delta} \bar{R}_{\delta\alpha\beta\gamma} h_{ip}^{\delta} + \sum_q \bar{R}_{i\alpha\beta\gamma} h_{qp}^{\alpha} \\
&\quad + \sum_q \bar{R}_{i\alpha q\gamma} h_{qp}^{\beta} + \sum_q \bar{R}_{i\alpha\beta q} h_{qp}^{\gamma}, \quad (3-186)
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{\alpha\beta\gamma\delta; p} &= \bar{R}_{\alpha\beta\gamma\delta, p} + \sum_q \bar{R}_{q\beta\gamma\delta} h_{qp}^{\alpha} + \sum_q \bar{R}_{\alpha q\gamma\delta} h_{qp}^{\beta} \\
&\quad + \sum_q \bar{R}_{\alpha\beta q\delta} h_{qp}^{\gamma} + \sum_q \bar{R}_{\alpha\beta\gamma q} h_{qp}^{\delta}. \quad (3-187)
\end{aligned}$$

**注释 3.4:** 特别注意, 符号  $\bar{R}_{ABCD; E}$  表示张量  $\bar{R}_{ABCD}$  在流形  $N$  上的协变导数, 而  $\bar{R}_{ABCD, i}$  表示张量  $\bar{R}_{ABCD}$  在流形  $M$  的拉回从  $x^*TN$  上的协变导数, 其意义是不一样的。

设  $N$  是空间形式  $R^{n+p}(c)$ , 则我们知道黎曼曲率满足如下的等式

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

将上面的关系式代入定理 3.3, 我们有如下的推论

**推论 3.2:** 设  $x: M \rightarrow R^{n+p}(c)$  是子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 则

$$\frac{\partial \theta^i}{\partial t} = \sum_j (V_{,j}^i - \sum_\alpha h_{ij}^\alpha V^\alpha - L_j^i) \theta^j, \quad \frac{\partial dv}{\partial t} = (\operatorname{div} V^i - n \sum_\alpha H^\alpha V^\alpha) dv, \quad (3-188)$$

$$\frac{\partial h_{ij}^\alpha}{\partial t} = V_{,ij}^\alpha + \sum_p h_{ij,p}^\alpha V^p + \sum_p h_{pj}^\alpha L_i^p + \sum_p h_{ip}^\alpha L_j^p - \sum_\beta h_{ij}^\beta L_\beta^\alpha + \sum_{p\beta} h_{ip}^\alpha h_{pj}^\beta V^\beta + c \delta_{ij} V^\alpha. \quad (3-189)$$

**推论 3.3:** 设  $x: M \rightarrow R^{n+1}(c)$  是超曲面,  $V = V^i e_i + fN$  是变分向量场, 则

$$\frac{\partial \theta^i}{\partial t} = \sum_j (V_{,j}^i - h_{ij} f - L_j^i) \theta^j, \quad \frac{\partial dv}{\partial t} = (\operatorname{div} V^i - n H f) dv, \quad (3-190)$$

$$\frac{\partial h_{ij}}{\partial t} = f_{,ij} + \sum_p h_{ij,p} V^p + \sum_p h_{pj} L_i^p + \sum_p h_{ip} L_j^p + \sum_p h_{ip} h_{pj} f + c \delta_{ij} f. \quad (3-191)$$

上面我们给出了第二基本型和余标架的变分公式, 对于由第二基本型组合而成的其他典型张量, 我们可以给出变分公式, 这些公式在后文之中大有用处。

**推论 3.4:** 设  $x: M \rightarrow N^{n+p}$  是子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 则

$$\frac{\partial S}{\partial t} = \sum 2h_{ij}^\alpha V_{,ij}^\alpha + \sum_i S_{,i} V^i + \sum 2S_{\alpha\alpha\beta} V^\beta - \sum 2h_{ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta, \quad (3-192)$$

$$\frac{\partial}{\partial t} H^\alpha = \frac{1}{n} \Delta V^\alpha + \sum_i H_{,i}^\alpha V^i - H^\beta L_\beta^\alpha + \frac{1}{n} S_{\alpha\beta} V^\beta + \frac{1}{n} \bar{R}_{\alpha\beta}^\gamma V^\beta, \quad (3-193)$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \sum_{ij\alpha} 2h_{ij}^\alpha V_{,ij}^\alpha - \sum_\alpha 2H^\alpha \Delta V^\alpha + \sum_i \rho_{,i} V^i + \sum_{\alpha\beta} 2(S_{\alpha\alpha\beta} - S_{\alpha\beta} H^\alpha) V^\beta \\ & - \sum_{ij\alpha\beta} 2h_{ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta - \sum_{\alpha\beta} 2H^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta, \end{aligned} \quad (3-194)$$

$$\begin{aligned} \frac{\partial S_{\alpha\beta}}{\partial t} = & V_{,ij}^\alpha h_{ij}^\beta + h_{ij}^\alpha V_{,ij}^\beta + S_{\alpha\beta,i} V^i + S_{\alpha\gamma} L_\beta^\gamma + S_{\beta\gamma} L_\alpha^\gamma \\ & + 2S_{\alpha\beta\gamma} V^\gamma - (\bar{R}_{ij\gamma}^\alpha h_{ij}^\beta + h_{ij}^\alpha \bar{R}_{ij\gamma}^\beta) V^\gamma, \end{aligned} \quad (3-195)$$

$$\begin{aligned} \frac{\partial S_{\alpha\beta\gamma}}{\partial t} = & V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\gamma + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\gamma + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\gamma \\ & + S_{\alpha\beta\gamma,i} V^i + S_{\gamma\beta\alpha} L_\alpha^\gamma + S_{\alpha\gamma\beta} V^\gamma + S_{\alpha\beta\gamma\delta} V^\delta + S_{\alpha\beta\gamma\delta} V^\delta \\ & - (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\gamma + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\gamma + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\gamma) V^\gamma, \end{aligned} \quad (3-196)$$

$$\begin{aligned} \frac{\partial \bar{R}_{ij\alpha}}{\partial t} = & \sum_\gamma \bar{R}_{ij\alpha;\gamma} V^\gamma + \sum_p \bar{R}_{ij\alpha;p} V^p + \sum_q \bar{R}_{q\beta\alpha} L_i^q + \sum_\gamma \bar{R}_{\gamma\beta\alpha} (V_{,i}^\gamma + h_{ip}^\gamma V^p) \\ & - \sum_q \bar{R}_{iq\alpha} (V_{,q}^\beta + h_{qp}^\beta V^p) + \sum_\gamma \bar{R}_{i\gamma\alpha} L_\beta^\gamma + \sum_q \bar{R}_{i\beta q\alpha} L_j^q + \sum_\gamma \bar{R}_{i\beta\gamma\alpha} (V_{,j}^\gamma + h_{jp}^\gamma V^p) \\ & - \sum_q \bar{R}_{i\beta q\alpha} (V_{,q}^\alpha + h_{qp}^\alpha V^p) + \sum_\gamma \bar{R}_{i\beta\gamma\alpha} L_\alpha^\gamma \end{aligned} \quad (3-197)$$

$$\begin{aligned}
&= \sum_{\gamma} \bar{R}_{i\beta\gamma\alpha} V^{\gamma} + \sum_p (\bar{R}_{i\beta\gamma\alpha, p} - \sum_{\gamma} \bar{R}_{\gamma\beta\gamma\alpha} h_{ip}^{\gamma} + \sum_q \bar{R}_{i\gamma\gamma\alpha} h_{qp}^{\beta} \\
&\quad - \sum_{\gamma} \bar{R}_{i\beta\gamma\alpha} h_{jp}^{\gamma} + \sum_q \bar{R}_{i\beta\gamma q} h_{qp}^{\alpha}) V^p \\
&\quad + \sum_q \bar{R}_{q\beta\gamma\alpha} L_i^q + \sum_{\gamma} \bar{R}_{\gamma\beta\gamma\alpha} (V_{,i}^{\gamma} + h_{ip}^{\gamma} V^p) \\
&\quad + \sum_q \bar{R}_{i\beta q\alpha} L_j^q + \sum_{\gamma} \bar{R}_{i\beta\gamma\alpha} (V_{,j}^{\gamma} + h_{jp}^{\gamma} V^p) \\
&\quad - \sum_q \bar{R}_{i\beta\gamma q} (V_{,q}^{\alpha} + h_{qp}^{\alpha} V^p) + \sum_{\gamma} \bar{R}_{i\beta\gamma\gamma} L_{\alpha}^{\gamma} \\
&\quad - \sum_q \bar{R}_{i\gamma q\alpha} (V_{,q}^{\beta} + h_{qp}^{\beta} V^p) + \sum_{\gamma} \bar{R}_{i\gamma\gamma\alpha} L_{\beta}^{\gamma}, \tag{3-198}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \bar{R}_{\alpha\beta}^{\gamma}}{\partial t} &= \sum_{i\gamma} \bar{R}_{\alpha i\beta\gamma} V^{\gamma} + \sum_{ip} \bar{R}_{\alpha i\beta ip} V^p \\
&\quad - \sum_{iq} \bar{R}_{q i\beta} (V_{,q}^{\alpha} + h_{qp}^{\alpha} V^p) + \sum_{i\gamma} \bar{R}_{\gamma i\beta} L_{\alpha}^{\gamma} \\
&\quad + \sum_{i\gamma} (\bar{R}_{\alpha\gamma i\beta} + \bar{R}_{\alpha i\gamma\beta}) (V_{,i}^{\gamma} + h_{ip}^{\gamma} V^p) \\
&\quad - \sum_{iq} \bar{R}_{\alpha i i q} (V_{,q}^{\beta} + h_{qp}^{\beta} V^p) + \sum_{i\gamma} \bar{R}_{\alpha i i \gamma} L_{\beta}^{\gamma} \tag{3-199}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i\gamma} \bar{R}_{\alpha i\beta\gamma} V^{\gamma} + \sum_{ip} (\bar{R}_{\alpha i\beta\gamma, p} + \sum_q \bar{R}_{q i\beta} h_{qp}^{\alpha} - \sum_{\gamma} \bar{R}_{\alpha\gamma i\beta} h_{ip}^{\gamma} \\
&\quad - \sum_{\gamma} \bar{R}_{\alpha i\gamma\beta} h_{ip}^{\gamma} + \sum_q \bar{R}_{\alpha i i q} h_{qp}^{\beta}) V^p \\
&\quad - \sum_{iq} \bar{R}_{q i\beta} (V_{,q}^{\alpha} + h_{qp}^{\alpha} V^p) + \sum_{i\gamma} \bar{R}_{\gamma i\beta} L_{\alpha}^{\gamma} \\
&\quad + \sum_{i\gamma} (\bar{R}_{\alpha\gamma i\beta} + \bar{R}_{\alpha i\gamma\beta}) (V_{,i}^{\gamma} + h_{ip}^{\gamma} V^p) \\
&\quad - \sum_{iq} \bar{R}_{\alpha i i q} (V_{,q}^{\beta} + h_{qp}^{\beta} V^p) + \sum_{i\gamma} \bar{R}_{\alpha i i \gamma} L_{\beta}^{\gamma}. \tag{3-200}
\end{aligned}$$

**推论 3.5:** 设  $x: M \rightarrow N^{n+1}(c)$  是超曲面,  $V = V^i e_i + fN$  是变分向量场, 则

$$\frac{\partial S}{\partial t} = \sum 2h_{ij} f_{,ij} + \sum S_{,i} V^i + 2P_3 f + \sum 2h_{ij} \bar{R}_{i(n+1)(n+1)j} f, \tag{3-201}$$

$$\frac{\partial H}{\partial t} = \frac{1}{n} (\Delta f + \sum_i n H_{,i} V^i + S f + \bar{R}_{(n+1)(n+1)} f), \tag{3-202}$$

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \sum_{ij} 2h_{ij} f_{,ij} - 2H \Delta f + \sum_i \rho_{,i} V^i + 2(P_3 - HS) f \\
&\quad + \sum_{ij} 2h_{ij} \bar{R}_{i(n+1)(n+1)j} f - 2H \bar{R}_{(n+1)(n+1)} f. \tag{3-203}
\end{aligned}$$

**推论 3.6:** 设  $x: M \rightarrow R^{n+p}(c)$  是子流形,  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 则

$$\frac{\partial S}{\partial t} = \sum 2h_{ij} V_{,ij}^{\alpha} + \sum S_{,i} V^i + \sum_{\alpha\beta} 2S_{\alpha\alpha\beta} V^{\beta} + \sum_{\alpha} 2nc H^{\alpha} V^{\alpha}, \tag{3-204}$$



$$\frac{\partial H^\alpha}{\partial t} = \frac{1}{n}(\Delta V^\alpha + \sum_i n H_{,i}^\alpha V^i - H^\beta L_\beta^\alpha + \sum_\beta S_{\alpha\beta} V^\beta + nc V^\alpha), \quad (3-205)$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \sum_{ij\alpha} 2h_{ij}^\alpha V_{,ij}^\alpha - \sum_\alpha 2H^\alpha \Delta V^\alpha + \sum_i \rho_{,i} V^i \\ & + \sum_{\alpha\beta} 2(S_{\alpha\beta} - S_{\alpha\beta} H^\alpha) V^\beta. \end{aligned} \quad (3-206)$$

**推论 3.7:** 设  $x: M \rightarrow R^{n+1}(c)$  是超曲面,  $V = V^i e_i + fN$  是变分向量场, 则

$$\frac{\partial S}{\partial t} = \sum 2h_{ij} f_{,ij} + \sum S_{,i} V^i + \sum 2h_{ij} h_{ip} h_{pj} f + 2nc H f, \quad (3-207)$$

$$\frac{\partial H}{\partial t} = \frac{1}{n}(\Delta f + \sum_i n H_{,i} V^i + S f + cn f), \quad (3-208)$$

$$\frac{\partial \rho}{\partial t} = \sum_{ij} 2h_{ij} f_{,ij} - 2H \Delta f + \sum_i \rho_{,i} V^i + (2P_3 - 2HS) f. \quad (3-209)$$

我们知道, 空间形式  $R^{n+p}(c)$  中的结构方程是

$$dX = \sum_A \sigma^A s_A, \quad ds_i = \omega_i^A s_A - c \sigma^i X, \quad (3-210)$$

$$ds_\alpha = \omega_\alpha^j s_j + \omega_\alpha^\beta s_\beta. \quad (3-211)$$

通过拉回运算, 我们有

$$(d_M + dt \wedge \frac{\partial}{\partial t})x = (\theta^A + dt V^A) e_A = \theta^i e_i + dt V^A e_A, \quad (3-212)$$

$$d_M x = \theta^i e_i, \quad \frac{d}{dt} x = V^A e_A, \quad (3-213)$$

$$(d_M + dt \wedge \frac{\partial}{\partial t})e_i = (\phi_i^A + dt L_i^A) e_A - c(\theta^i + dt V^i) x, \quad (3-214)$$

$$d_M e_i = \phi_i^A e_A - c \theta^i x, \quad \frac{d}{dt} e_i = L_i^A e_A - c V^i x, \quad (3-215)$$

$$(d_M + dt \wedge \frac{\partial}{\partial t})e_\alpha = (\phi_\alpha^A + dt L_\alpha^A) e_A, \quad (3-216)$$

$$d_M e_\alpha = \phi_\alpha^A e_A, \quad \frac{d}{dt} e_\alpha = L_\alpha^A e_A. \quad (3-217)$$

**命题 3.2:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 设  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 则

$$\frac{d}{dt} x = V^A e_A, \quad \frac{d}{dt} e_i = L_i^A e_A - c V^i x, \quad \frac{d}{dt} e_\alpha = L_\alpha^A e_A.$$

**推论 3.8:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 设  $V = V^i e_i + fN$  是变分向量场, 则

$$\frac{d}{dt} x = V^i e_i + fN, \quad \frac{d}{dt} e_i = L_i^j e_j + L_i^{n+1} N - c V^i x, \quad \frac{d}{dt} N = L_{n+1}^i e_i.$$

## 第4章 张量的组合构造

本章主要研究广义 Newton 变换的定义和性质。

### 4.1 Newton 变换的定义

设  $x: M^n \rightarrow N^{n+p}$  是子流形,  $B$  是其第二基本型。记为

$$B = B_{ij}\theta^i\theta^j = (h_{ij}^\alpha e_\alpha)\theta^i\theta^j, \quad \hat{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha\delta_{ij}, \quad (4-1)$$

$$\dot{B} = \hat{h}_{ij}^\alpha\theta^i\otimes\theta^j\otimes e_\alpha = B - \bar{H}\otimes(ds^2), \quad (4-2)$$

$$\dot{B}_{ij} = (h_{ij}^\alpha - H^\alpha\delta_{ij})\otimes e_\alpha = B_{ij} - \delta_{ij}\bar{H}. \quad (4-3)$$

余维数  $p$  不同时, 我们分别讨论。

- $p=1$ , 即  $x$  是超曲面时, 固定法向量  $e_{n+1}$ , 则

$$B = h_{ij}\theta^i\theta^j, \quad \hat{h}_{ij} = h_{ij} - H\delta_{ij}.$$

- $p=1, 0 \leq r \leq n$ , 第  $r$  个曲率函数为

$$S_0 = 1, \quad S_r = \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} h_{i_1 j_1} \cdots h_{i_r j_r}, \quad (4-4)$$

$$H_0 = 1, \quad H_r = \frac{S_r}{\binom{n}{r}}, \quad H_1 = H, \quad (4-5)$$

$$\hat{S}_0 = 1, \quad \hat{S}_r = \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} \hat{h}_{i_1 j_1} \cdots \hat{h}_{i_r j_r} = \sum_{a=0}^r (-1)^a \binom{n+a-r}{a} (H)^a S_{r-a}, \quad (4-6)$$

$$\hat{H}_0 = 1, \quad \hat{H}_r = \sum_{a=0}^r (-1)^a \binom{r}{a} (H)^a H_{r-a}, \quad \hat{H}_1 = \hat{H} = 0. \quad (4-7)$$

- $p=1, 0 \leq r \leq n$ , 第  $r$  个经典 Newton 变换为

$$T_{(0)j}^i = \delta_j^i, \quad T_{(r)j}^i = \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} h_{i_1 j_1} \cdots h_{i_r j_r}, \quad (4-8)$$

$$\widehat{T}_{(0)j}^i = \delta_j^i, \quad \widehat{T}_{(r)j}^i = \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} \hat{h}_{i_1 j_1} \cdots \hat{h}_{i_r j_r}, \quad (4-9)$$

$$\widehat{T}_{(0)j}^i = \delta_j^i, \quad \widehat{T}_{(r)j}^i = \sum_{a=0}^r (-1)^a (H)^a \binom{n+a-r-1}{a} T_{(r-a)j}^i. \quad (4-10)$$

- $p=1, 0 \leq r, s \leq n$ , 第  $r$  个广义 Newton 变换为

$$T_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} = \frac{1}{r!} \delta_{j_1 \cdots j_r l_1 \cdots l_s}^{i_1 \cdots i_r k_1 \cdots k_s} h_{i_{j_1}} \cdots h_{i_{j_r}}, \quad (4-11)$$

$$\widehat{T}_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} = \frac{1}{r!} \delta_{j_1 \cdots j_r l_1 \cdots l_s}^{i_1 \cdots i_r k_1 \cdots k_s} \hat{h}_{i_{j_1}} \cdots \hat{h}_{i_{j_r}}, \quad (4-12)$$

$$\widehat{T}_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} = \sum_{a=0}^r (-1)^a (H)^a \binom{n+a-r-s}{a} T_{(r-a)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}. \quad (4-13)$$

对于超曲面的广义 Newton 变换, 我们有

- $s=0$ , 第  $r$  个广义 Newton 变换  $T_{(r)}$  为曲率函数  $S_r$ ;
- $s=1$ , 第  $r$  个广义 Newton 变换  $T_{(r)}$  为经典 Newton 变换  $T_{(r)j}^i$ ;
- $p \geq 2$ , 即  $x$  是高余维子流形时

$$B = B_{ij} \theta^i \theta^j = (h_{ij}^\alpha e_\alpha) \theta^i \theta^j, \quad \hat{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij}, \quad (4-14)$$

$$\hat{B} = \hat{h}_{ij}^\alpha \theta^i \otimes \theta^j \otimes e_\alpha = B - \bar{H} \otimes ds^2, \quad (4-15)$$

$$\hat{B}_{ij} = \hat{h}_{ij}^\alpha \otimes e_\alpha = \sum_{\alpha} (h_{ij}^\alpha - H^\alpha \delta_{ij}) e_\alpha = B_{ij} - \bar{H} \delta_{ij}, \quad (4-16)$$

$$\langle \hat{B}_{ij}, \hat{B}_{kl} \rangle = \langle B_{ij}, B_{kl} \rangle - \delta_{ij} \langle \bar{H}, B_{kl} \rangle - \delta_{kl} \langle \bar{H}, B_{ij} \rangle + \delta_{ij} \delta_{kl} H^2. \quad (4-17)$$

- $p \geq 2, r$  为偶数, 第  $r$  个经典 Newton 变换为

$$T_{(0)j}^i = \delta_{ij}, \quad (4-18)$$

$$T_{(r)j}^i = \frac{1}{r!} \delta_{j_1 \cdots j_r j}^{i_1 \cdots i_r i} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle, \quad (4-19)$$

$$\widehat{T}_{(0)j}^i = \delta_j^i, \quad (4-20)$$

$$\widehat{T}_{(r)j}^i = \frac{1}{r!} \delta_{j_1 \cdots j_r j}^{i_1 \cdots i_r i} \langle \hat{B}_{i_{j_1}}, \hat{B}_{i_{j_2}} \rangle \cdots \langle \hat{B}_{i_{j_{r-1}}}, \hat{B}_{i_{j_r}} \rangle. \quad (4-21)$$

- $p \geq 2, r$  为奇数, 第  $r$  个经典 Newton 变换为

$$T_{(r)\bar{y}}^\alpha = \frac{1}{(r)!} \delta_{j_1 \cdots j_r j}^{i_1 \cdots i_r i} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-2j_r-2}}}, B_{i_{j_{r-1j_r-1}}} \rangle h_{i_{j_r}}^\alpha, \quad (4-22)$$

$$\widehat{T}_{(r)\bar{y}}^\alpha = \frac{1}{(r)!} \delta_{j_1 \cdots j_r j}^{i_1 \cdots i_r i} \langle \hat{B}_{i_{j_1}}, \hat{B}_{i_{j_2}} \rangle \cdots \langle \hat{B}_{i_{j_{r-2j_r-2}}}, \hat{B}_{i_{j_{r-1j_r-1}}} \rangle \hat{h}_{i_{j_r}}^\alpha. \quad (4-23)$$

- $p \geq 2, r$  为偶数, 第  $r$  个曲率函数为

$$\begin{aligned} S_r &= \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ &= \sum_{i\alpha} \frac{1}{r} T_{(r-1)\bar{y}}^\alpha h_{i\bar{y}}^\alpha, \end{aligned} \quad (4-24)$$

$$S_0 = 1, H_r = \frac{S_r}{\binom{n}{r}}, \quad (4-25)$$

$$\begin{aligned}\hat{S}_r &= \frac{1}{r!} \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \langle \hat{B}_{i_{j_1}}, \hat{B}_{i_{j_2}} \rangle \dots \langle \hat{B}_{i_{j_{r-1}}}, \hat{B}_{i_{j_r}} \rangle \\ &= \sum_{j\alpha} \frac{1}{r} T_{(r-1)j}^{\alpha} \hat{h}_{j\alpha}^{\alpha},\end{aligned}\quad (4-26)$$

$$\hat{S}_0 = 1, \quad \hat{H}_r = \frac{\hat{S}_r}{\binom{n}{r}}. \quad (4-27)$$

•  $p \geq 2$ ,  $r$  为奇数, 第  $r$  个曲率向量为

$$\bar{S}_r = \frac{1}{(r)!} \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \times \langle B_{i_{j_{r-2}}}, B_{i_{j_{r-1}}} \rangle B_{i_{j_r}}, \quad (4-28)$$

$$= \sum_{j\alpha} \frac{1}{r} T_{(r-1)j}^{\alpha} h_{j\alpha}^{\alpha} e_{\alpha} = : S_r^{\alpha} e_{\alpha}, \quad (4-29)$$

$$\bar{H}_r = \frac{\bar{S}_r}{\binom{n}{r}} = : H_r^{\alpha} e_{\alpha}, \quad (4-30)$$

$$\widehat{\bar{S}}_r = \frac{1}{(r)!} \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \langle \hat{B}_{i_{j_1}}, \hat{B}_{i_{j_2}} \rangle \dots \times \langle \hat{B}_{i_{j_{r-2}}}, \hat{B}_{i_{j_{r-1}}} \rangle \hat{B}_{i_{j_r}}, \quad (4-31)$$

$$= \sum_{j\alpha} \frac{1}{r} T_{(r-1)j}^{\alpha} \hat{h}_{j\alpha}^{\alpha} e_{\alpha} = : \hat{S}_r^{\alpha} e_{\alpha}, \quad (4-32)$$

$$\widehat{\bar{H}}_r = \frac{\widehat{\bar{S}}_r}{\binom{n}{r}} = : \hat{H}_r^{\alpha} e_{\alpha}. \quad (4-33)$$

•  $p \geq 2$ ,  $r$  为偶数,  $t, s \in N$ , 广义 Newton 变换为

$$\begin{aligned}T_{(r,t)}^{\alpha_1 \dots \alpha_t}_{k_1 \dots k_s; l_1 \dots l_s} &= \frac{1}{(r+t)!} \delta_{j_1 \dots j_{r+t}}^{i_1 \dots i_{r+t}} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \\ &\quad \times \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle h_{p_1 q_1}^{\alpha_1} \dots h_{p_t q_t}^{\alpha_t},\end{aligned}\quad (4-34)$$

$$\begin{aligned}\widehat{T}_{(r,t)}^{\alpha_1 \dots \alpha_t}_{k_1 \dots k_s; l_1 \dots l_s} &= \frac{1}{(r+t)!} \delta_{j_1 \dots j_{r+t}}^{i_1 \dots i_{r+t}} \langle \hat{B}_{i_{j_1}}, \hat{B}_{i_{j_2}} \rangle \dots \\ &\quad \times \langle \hat{B}_{i_{j_{r-1}}}, \hat{B}_{i_{j_r}} \rangle \hat{h}_{p_1 q_1}^{\alpha_1} \dots \hat{h}_{p_t q_t}^{\alpha_t}.\end{aligned}\quad (4-35)$$

对于余维数大于等于 2 的子流形的广义 Newton 变换, 有

•  $p \geq 2, s = 0$

$$T_{(r,t)}^{\alpha_1 \dots \alpha_t}_{k_1 \dots k_s; l_1 \dots l_s} = : T_{(r)}^{\alpha_1 \dots \alpha_t}, \quad (4-36)$$

$$\widehat{T}_{(r,t)}^{\alpha_1 \dots \alpha_t}_{k_1 \dots k_s; l_1 \dots l_s} = : \widehat{T}_{(r)}^{\alpha_1 \dots \alpha_t}. \quad (4-37)$$

•  $p \geq 2, t = 0$

$$T_{(r,0)}^{\alpha_1 \dots \alpha_t}_{k_1 \dots k_s; l_1 \dots l_s} = : T_{(r)}^{\alpha_1 \dots \alpha_t}, \quad (4-38)$$

$$\widehat{T}_{(r,0)k_1 \cdots k_s; l_1 \cdots l_s} = : \widehat{T}_{(r)l_1 \cdots l_s}^{k_1 \cdots k_s}. \quad (4-39)$$

$$\bullet \quad p \geq 2, r=0, t=0$$

$$T_{(0,0)k_1 \cdots k_s; l_1 \cdots l_s} = \delta_{l_1 \cdots l_s}^{k_1 \cdots k_s}, \quad (4-40)$$

$$\widehat{T}_{(0,0)k_1 \cdots k_s; l_1 \cdots l_s} = \delta_{l_1 \cdots l_s}^{k_1 \cdots k_s}. \quad (4-41)$$

$$\bullet \quad p \geq 2, t=1, s=1$$

$$\begin{aligned} T_{(r,1)k;l}^\alpha &= \frac{1}{(r+1)!} \delta_{j_1 \cdots j_{j_r+1}^l}^{i_1 \cdots i_{j_r+1}^k} \langle B_{ij_1}, B_{ij_2} \rangle \cdots \\ &\quad \langle B_{i_{r-j_r-1}}, B_{ij_r} \rangle h_{i_r+j_r+1}^\alpha = T_{(r+1)kl}^\alpha, \end{aligned} \quad (4-42)$$

$$\begin{aligned} \widehat{T}_{(r,1)k;l}^\alpha &= \frac{1}{(r+1)!} \delta_{j_1 \cdots j_{j_r+1}^l}^{i_1 \cdots i_{j_r+1}^k} \langle \hat{B}_{ij_1}, \hat{B}_{ij_2} \rangle \cdots \\ &\quad \langle \hat{B}_{i_{r-j_r-1}}, \hat{B}_{ij_r} \rangle \hat{h}_{i_r+j_r+1}^\alpha = \widehat{T}_{(r+1)kl}^\alpha. \end{aligned} \quad (4-43)$$

$$\bullet \quad p \geq 2, t=0, s=0$$

$$T_{(r,0)} = \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_r-1}}, B_{ij_r} \rangle = S_r, \quad (4-44)$$

$$\widehat{T}_{(r,0)} = \frac{1}{r!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} \langle \hat{B}_{ij_1}, \hat{B}_{ij_2} \rangle \cdots \langle \hat{B}_{i_{r-j_r-1}}, \hat{B}_{ij_r} \rangle = \hat{S}_r. \quad (4-45)$$

$$\bullet \quad p \geq 2, t=0, s=1$$

$$T_{(r,0)k;l}^P = \frac{1}{(r)!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r k} \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_r-1}}, B_{ij_r} \rangle = T_{(r)l}^k, \quad (4-46)$$

$$\begin{aligned} \widehat{T}_{(r,0)k;l} &= \frac{1}{(r)!} \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r k} \langle \hat{B}_{ij_1}, \hat{B}_{ij_2} \rangle \cdots \\ &\quad \times \langle \hat{B}_{i_{r-j_r-1}}, \hat{B}_{ij_r} \rangle = \widehat{T}_{(r)l}^k = \widehat{T}_{(r)kl}. \end{aligned} \quad (4-47)$$

$$\bullet \quad p \geq 2 \text{ 时, } r+t+s > n$$

$$T_{(r,t)k_1 \cdots k_s; l_1 \cdots l_s}^{\alpha_1 \cdots \alpha_t} = \frac{1}{(r+t)!} \delta_{j_1 \cdots j_r q_1 \cdots q_t}^{i_1 \cdots i_r p_1 \cdots p_t k_1 \cdots k_s} = 0, \quad (4-48)$$

$$\widehat{T}_{(r,t)k_1 \cdots k_s; l_1 \cdots l_s}^{\alpha_1 \cdots \alpha_t} = \frac{1}{(r+t)!} \delta_{j_1 \cdots j_r q_1 \cdots q_t}^{i_1 \cdots i_r p_1 \cdots p_t k_1 \cdots k_s} = 0. \quad (4-49)$$

## 4.2 Newton 变换的性质

本节主要研究 Newton 变换的性质。

对于广义的 Kronecker 符号, 我们有下面的重要性质——行列式刻画。

$$\delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} = \begin{pmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_1}^{i_r} \\ \vdots & & \vdots \\ \delta_{j_r}^{i_1} & \cdots & \delta_{j_r}^{i_r} \end{pmatrix}$$

引理 4.1 ( $\delta$  性质): 我们有如下等式:

$$\delta_{\dots ij \dots} = -\delta_{\dots ji \dots}, \quad (4-50)$$

$$\delta_{\dots ij \dots} = -\delta_{\dots ji \dots}, \quad (4-51)$$

$$\delta_{j_1 \dots j_r}^{i_1 \dots i_r} = \delta_{i_1 \dots i_r}^{j_1 \dots j_r}, \quad (4-52)$$

$$\sum_p \delta_{j_1 \dots j_p}^{i_1 \dots i_p} = (n-r) \delta_{j_1 \dots j_r}^{i_1 \dots i_r}, \quad (4-53)$$

$$\delta_{j_a \dots j_b}^{i_a \dots i_b} = \delta_{j_b \dots j_a}^{i_b \dots i_a}, \quad (4-54)$$

$$\delta_{j_1 \dots j_d}^{i_1 \dots i_d} = \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \delta_j^i - \sum_{a=1}^r \delta_{j_1 \dots j_a-1 j_a+1 \dots j_d}^{i_1 \dots i_a-1 i_a+1 \dots i_d} \delta_j^{i_a}, \quad (4-55)$$

$$= \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \delta_j^i - \sum_{a=1}^r \delta_{j_1 \dots j_a-1 j_a+1 \dots j_d}^{i_1 \dots i_a-1 i_a+1 \dots i_d} \delta_{j_a}^{i_a}. \quad (4-56)$$

证明: 由广义 Kronecker 符号的行列式刻画, 有

$$\delta_{j_1 \dots j_d}^{i_1 \dots i_d} = \delta_{j_1 \dots j_r}^{i_1 \dots i_r} = \det \begin{pmatrix} \delta_j^i & \delta_{j_1}^i & \dots & \delta_{j_r}^i \\ \delta_j^{i_1} & \delta_{j_1}^{i_1} & \dots & \delta_{j_r}^{i_1} \\ \vdots & \vdots & & \vdots \\ \delta_j^{i_r} & \delta_{j_1}^{i_r} & \dots & \delta_{j_r}^{i_r} \end{pmatrix} \quad (4-57)$$

$$= \delta_j^i \delta_{j_1 \dots j_r}^{i_1 \dots i_r} - \delta_j^{i_1} \delta_{j_2 \dots j_r}^{i_2 \dots i_r} + \dots + (-1)^r \delta_j^{i_r} \delta_{j_1 \dots j_{r-1}}^{i_1 \dots i_{r-1}} \quad (4-58)$$

$$= \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \delta_j^i - \sum_{a=1}^r \delta_{j_1 \dots j_a-1 j_a+1 \dots j_d}^{i_1 \dots i_a-1 i_a+1 \dots i_d} \delta_j^{i_a} \quad (4-59)$$

$$= \delta_{j_1 \dots j_r}^{i_1 \dots i_r} \delta_j^i - \sum_{a=1}^r \delta_{j_1 \dots j_a-1 j_a+1 \dots j_d}^{i_1 \dots i_a-1 i_a+1 \dots i_d} \delta_{j_a}^{i_a}. \quad (4-60)$$

命题 4.1 (对称性): 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么我们有

$$T_{(r,t)}^{a_1 \dots a_t}_{k_1 \dots k_s; l_1 \dots l_s} = T_{(r,t)}^{a_1 \dots a_t}_{l_1 \dots l_s; k_1 \dots k_s}, \quad (4-61)$$

$$T_{(r,t)}^{a_1 \dots a_r}_{k_1 \dots k_s; l_1 \dots l_s} = T_{(r,t)}^{a_1 \dots a_r}_{k_1 \dots k_s; l_1 \dots l_s}, \quad (4-62)$$

$$\widehat{T}_{(r,t)}^{a_1 \dots a_t}_{k_1 \dots k_s; l_1 \dots l_s} = \widehat{T}_{(r,t)}^{a_1 \dots a_t}_{l_1 \dots l_s; k_1 \dots k_s}, \quad (4-63)$$

$$\widehat{T}_{(r,t)}^{a_1 \dots a_r}_{k_1 \dots k_s; l_1 \dots l_s} = \widehat{T}_{(r,t)}^{a_1 \dots a_r}_{k_1 \dots k_s; l_1 \dots l_s}. \quad (4-64)$$

证明: 对于第一式(式 4-61), 由引理 4.1 和第二基本型的对称性, 我们有

$$\begin{aligned} & T_{(r,t)}^{a_1 \dots a_t}_{k_1 \dots k_s; l_1 \dots l_s} \\ &= \frac{1}{(r+t)!} \delta_{j_1 \dots j_{r+1} \dots j_{r+t}}^{i_1 \dots i_r \dots i_{r+t}} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ & \quad \times (h_{p_1 q_1}^{a_1} \dots h_{p_t q_t}^{a_t}) \end{aligned} \quad (4-65)$$

$$\begin{aligned} &= \frac{1}{(r+t)!} \delta_{i_1 \dots i_r \dots i_{r+t}}^{j_1 \dots j_{r+1} \dots j_{r+t}} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ & \quad \times (h_{p_1 q_1}^{a_1} \dots h_{p_t q_t}^{a_t}) \end{aligned} \quad (4-66)$$

$$\begin{aligned} &= \frac{1}{(r+t)!} \delta_{i_1 \dots i_r \dots i_{r+t}}^{j_1 \dots j_{r+1} \dots j_{r+t}} \langle B_{j_1 i_1}, B_{j_2 i_2} \rangle \dots \langle B_{j_{r-1} i_{r-1}}, B_{j_r i_r} \rangle \\ & \quad \times (h_{q_1 p_1}^{a_1} \dots h_{q_t p_t}^{a_t}) \end{aligned} \quad (4-67)$$

$$= \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t k_1 \dots k_s}^{i_1 \dots i_r p_1 \dots p_t l_1 \dots l_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \times (h_{p_1 q_1}^{\alpha_1} \dots h_{p_t q_t}^{\alpha_t}) \quad (4-68)$$

$$= T_{(r,t) l_1 \dots l_s; k_1 \dots k_s}^{\alpha_1 \dots \alpha_t} \quad (4-69)$$

对于第二式(式4-62), 同样地

$$T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \\ = \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t q_j' \dots q_j' l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_t k_1 \dots k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \dots h_{p_{q_i}}^{\alpha_i} \dots h_{p_{q_j}}^{\alpha_j} \dots \quad (4-70)$$

$$= \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t q_j' \dots q_j' l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_t k_1 \dots k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \dots h_{p_{q_i}}^{\alpha_i} \dots h_{p_{q_j}}^{\alpha_j} \dots \quad (4-71)$$

$$= \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t q_j' \dots q_j' l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_t k_1 \dots k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \dots h_{p_{q_j}}^{\alpha_j} \dots h_{p_{q_i}}^{\alpha_i} \dots \quad (4-72)$$

$$= \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t q_j' \dots q_j' l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_t k_1 \dots k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \dots h_{p_{q_i}}^{\alpha_i} \dots h_{p_{q_j}}^{\alpha_j} \dots \quad (4-73)$$

$$= T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \quad (4-74)$$

**命题 4.2 (反对称性):** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么我们有

$$T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} = -T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \quad (4-75)$$

$$T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} = -T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \quad (4-76)$$

$$\widehat{T}_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} = -\widehat{T}_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \quad (4-77)$$

$$\widehat{T}_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} = -\widehat{T}_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \quad (4-78)$$

**证明:** 由引理 4.1, 我们有

$$T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \\ = \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t q_j' \dots q_j' l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_t k_1 \dots k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \dots h_{p_{q_i}}^{\alpha_i} \dots h_{p_{q_j}}^{\alpha_j} \dots \quad (4-79)$$

$$= -\frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_t q_j' \dots q_j' l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_t k_1 \dots k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ \dots h_{p_{q_i}}^{\alpha_i} \dots h_{p_{q_j}}^{\alpha_j} \dots \quad (4-80)$$

$$= -T_{(r,t) k_1 \dots k_s; l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \quad (4-81)$$

**命题 4.3 (迹性质):** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么我们有

$$\sum_{k_s} T_{(r,t) k_1 \dots k_{s-1} k_s; l_1 \dots l_{s-1} l_s}^{\alpha_1 \dots \alpha_t} = (n+1-r-t-s) T_{(r,t) k_1 \dots k_{s-1} l_1 \dots l_{s-1}}^{\alpha_1 \dots \alpha_t} \quad (4-82)$$

$$\sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_t-2\beta\beta}_{k_1 \cdots k_s; l_1 \cdots l_s} = T_{(r+2, t-2)}^{\alpha_1 \cdots \alpha_t-2}_{k_1 \cdots k_s; l_1 \cdots l_s}, \quad (4-83)$$

$$\sum_{k_s} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s-1} k_s; l_1 \cdots l_{s-1} l_s} = (n+1-r-t-s) \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s-1}; l_1 \cdots l_{s-1}}, \quad (4-84)$$

$$\sum_{\beta} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t-2\beta\beta}_{k_1 \cdots k_s; l_1 \cdots l_s} = \widehat{T}_{(r+2, t-2)}^{\alpha_1 \cdots \alpha_t-2}_{k_1 \cdots k_s; l_1 \cdots l_s}. \quad (4-85)$$

证明: 对于第一式(4-82), 由引理4.1, 有

$$\begin{aligned} & \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s-1} k_s; l_1 \cdots l_{s-1} l_s} \\ &= \sum_{k_s} \frac{1}{(r+t)!} \delta_{j_1 \cdots j_{\rho_1} \cdots p, k_1 \cdots k_{s-1} k_s}^{i_1 \cdots i_{\rho_1} \cdots p} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ & \quad \times (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_{\rho_1}}^{\alpha_t}) \end{aligned} \quad (4-86)$$

$$\begin{aligned} &= \sum_p \frac{1}{(r+t)!} \delta_{j_1 \cdots j_{\rho_1} \cdots q, d_1 \cdots d_{s-1} d_s}^{i_1 \cdots i_{\rho_1} \cdots p, k_1 \cdots k_{s-1} k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ & \quad \times (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_{\rho_1}}^{\alpha_t}) \end{aligned} \quad (4-87)$$

$$= (n+1-r-t-s) T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s-1}; l_1 \cdots l_{s-1}}. \quad (4-88)$$

对于(4-83), 由定义

$$\begin{aligned} & \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_t-2\beta\beta}_{k_1 \cdots k_{s-1} k_s; l_1 \cdots l_{s-1} l_s} \\ &= \sum_{\beta} \frac{1}{(r+t)!} \delta_{j_1 \cdots j_{\rho_1} \cdots q, d_1 \cdots d_{s-1} d_s}^{i_1 \cdots i_{\rho_1} \cdots p, k_1 \cdots k_{s-1} k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ & \quad \times (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_{t-2q_t-2}}^{\alpha_{t-2}} h_{p_{t-1q_{t-1}}}^{\beta} h_{p_{\rho_1}}^{\beta}) \end{aligned} \quad (4-89)$$

$$\begin{aligned} &= \sum_{\beta} \frac{1}{(r+t)!} \delta_{j_1 \cdots j_{\rho_1} \cdots q, d_1 \cdots d_{s-1} d_s}^{i_1 \cdots i_{\rho_1} \cdots p, k_1 \cdots k_{s-1} k_s} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \cdots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle \\ & \quad \times (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_{t-2q_t-2}}^{\alpha_{t-2}}) \langle B_{p_{t-1q_{t-1}}}, B_{p_{\rho_1}} \rangle \end{aligned} \quad (4-90)$$

$$= T_{(r+2, t-2)}^{\alpha_1 \cdots \alpha_t-2}_{k_1 \cdots k_s; l_1 \cdots l_s}. \quad (4-91)$$

**命题4.4**(协变导数): 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 对于 Newton 变换的协变导数有

$$\begin{aligned} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, p} &= \sum_{ij} \sum_{\beta} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{\beta} \beta}_{k_1 \cdots k_s; l_1 \cdots l_s, ij} h_{ij, p}^{\beta} \\ & \quad + \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, ij} h_{ij, p}^{\alpha_b}, \end{aligned} \quad (4-92)$$

$$\begin{aligned} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, p} &= \sum_{ij} \sum_{\beta} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{\beta} \beta}_{k_1 \cdots k_s; l_1 \cdots l_s, ij} \hat{h}_{ij, p}^{\beta} \\ & \quad + \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, ij} \hat{h}_{ij, p}^{\alpha_b}, \end{aligned} \quad (4-93)$$

证明: 由引理4.1和定义, 有



$$\begin{aligned}
T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, p} &= \frac{1}{(r+t)!} \delta_{j_1 \cdots j_r q_1 \cdots q_t}^{i_1 \cdots i_r p_1 \cdots p_t k_1 \cdots k_s} \times [ \langle B_{ij_1, p}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_{r-1}}, B_{ij_r}} \rangle \\
&\quad + \langle B_{ij_1}, B_{ij_2, p} \rangle \cdots \langle B_{i_{r-j_{r-1}}, B_{ij_r}} \rangle \\
&\quad + \cdots + \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_{r-1}}, p}, B_{ij_r} \rangle \\
&\quad + \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_{r-1}}, B_{ij_r, p}} \rangle ] \times (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_t q_t}^{\alpha_t}) \\
&\quad + \frac{1}{(r+t)!} \delta_{j_1 \cdots j_r q_1 \cdots q_t}^{i_1 \cdots i_r p_1 \cdots p_t k_1 \cdots k_s} \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_{r-1}}, B_{ij_r}} \rangle \\
&\quad \times \left( \sum_{b=1}^t h_{p_1 q_1}^{\alpha_1} \cdots h_{p_b q_b, p}^{\alpha_b} \cdots h_{p_t q_t}^{\alpha_t} \right) \quad (4-94)
\end{aligned}$$

$$\begin{aligned}
&= \frac{r}{(r+t)!} \delta_{j_1 \cdots j_r q_1 \cdots q_t}^{i_1 \cdots i_r p_1 \cdots p_t k_1 \cdots k_s} \times \\
&\quad \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-2j_{r-3}}, B_{i_{r-2j_{r-2}}}} \rangle \left( \sum_{\alpha_{t+1}} h_{i_{r-j_{r-1}} p}^{\alpha_{t+1}} h_{ij_r, p}^{\alpha_{t+1}} \right) \\
&\quad \times (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_t q_t}^{\alpha_t}) \\
&\quad + \frac{1}{(r+t)!} \delta_{j_1 \cdots j_r q_1 \cdots q_t}^{i_1 \cdots i_r p_1 \cdots p_t k_1 \cdots k_s} \langle B_{ij_1}, B_{ij_2} \rangle \cdots \langle B_{i_{r-j_{r-1}}, B_{ij_r}} \rangle \\
&\quad \times \left( \sum_{b=1}^t h_{p_1 q_1}^{\alpha_1} \cdots h_{p_b q_b, p}^{\alpha_b} \cdots h_{p_t q_t}^{\alpha_t} \right) \quad (4-95)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} h_{ij, p}^{\alpha_{t+1}} \\
&\quad + \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} h_{ij, p}^{\alpha_b} \quad (4-96)
\end{aligned}$$

特别的, 在命题 4.4 中令  $r$  是偶数和  $t=s=0$ , 我们有

**推论 4.1:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么我们有

•  $r$  为偶数

$$S_{r,p} = \sum_{ij\alpha} T_{(r-2, 1)}^{\alpha} h_{ij, p}^{\alpha} = \sum_{ij\alpha} T_{(r-1)}^{\alpha} h_{ij, p}^{\alpha}, \quad (4-97)$$

$$\hat{S}_{r,p} = \sum_{ij\alpha} \widehat{T_{(r-2, 1)}^{\alpha}} h_{ij, p}^{\alpha} = \sum_{ij\alpha} \widehat{T_{(r-1)}^{\alpha}} h_{ij, p}^{\alpha}, \quad (4-98)$$

$$T_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s, p} = \sum_{ij} \sum_{\alpha} T_{(r-2, 1)}^{\alpha} h_{ij, p}^{\alpha}, \quad (4-99)$$

$$\widehat{T_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s, p}} = \sum_{ij} \sum_{\alpha} \widehat{T_{(r-2, 1)}^{\alpha}} h_{ij, p}^{\alpha}. \quad (4-100)$$

•  $r$  为奇数

$$S_{r,p}^{\alpha} = \sum_{ij} \sum_{\beta} \frac{r-1}{r} T_{(r-3, 2)}^{\alpha\beta} h_{ij, p}^{\beta} + \sum_{ij} \frac{1}{r} T_{(r-1)}^{\alpha} h_{ij, p}^{\alpha}, \quad (4-101)$$

$$\hat{S}_{r,p}^{\alpha} = \sum_{ij} \sum_{\beta} \frac{r-1}{r} \widehat{T_{(r-3, 2)}^{\alpha\beta}} h_{ij, p}^{\beta} + \sum_{ij} \frac{1}{r} \widehat{T_{(r-1)}^{\alpha}} h_{ij, p}^{\alpha}, \quad (4-102)$$

**推论 4.2:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么我们有

$$S_{r,p} = \sum_{ij} T_{(r-2,1)}^{n+1}{}_{ij} h_{ij,p}^{n+1} = \sum_{ij} T_{(r-1)}^{i,j} h_{ij,p}^i, \quad (4-103)$$

$$T_{(r)}^{k_1 \cdots k_s}{}_{l_1 \cdots l_s, p} = \sum_{ij} \sum_{\alpha} T_{(r-1)}^{k_1 \cdots k_s i}{}_{l_1 \cdots l_s j} h_{ij,p}^{\alpha}, \quad (4-104)$$

$$\hat{S}_{r,p} = \sum_{ij} T_{(r-2,1)}^{n+1}{}_{ij} \hat{h}_{ij,p}^{n+1} = \sum_{ij} T_{(r-1)}^{i,j} \hat{h}_{ij,p}^i, \quad (4-105)$$

$$\widehat{T}_{(r)}^{k_1 \cdots k_s}{}_{l_1 \cdots l_s, p} = \sum_{ij} T_{(r-1)}^{k_1 \cdots k_s i}{}_{l_1 \cdots l_s j} \hat{h}_{ij,p}^i. \quad (4-106)$$

**命题 4.5** (散度性质): 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么我们有

$$\begin{aligned} \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}{}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= \frac{1}{2} \left( \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{jk_s}^{\alpha_{t+1}} \right. \\ &\quad \left. + \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{jk_s}^{\alpha_b} \right), \end{aligned} \quad (4-107)$$

$$\begin{aligned} \sum_{l_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}{}_{k_1 \cdots k_s; l_1 \cdots l_s, l_s} &= \frac{1}{2} \left( \sum_{l_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{il_j}^{\alpha_{t+1}} \right. \\ &\quad \left. + \sum_{l_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{il_j}^{\alpha_b} \right), \end{aligned} \quad (4-108)$$

$$\begin{aligned} \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}{}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= \frac{1}{2} \left( \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{jk_s}^{\alpha_{t+1}} \right. \\ &\quad \left. + \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{jk_s}^{\alpha_b} \right) \\ &\quad - \sum_{k_s \alpha_{t+1}} \frac{r(n+1-r-t-s)}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}{}_{k_1 \cdots k_s i; l_1 \cdots l_s} H_{k_s}^{\alpha_{t+1}} \\ &\quad - \sum_{k_s} \sum_{b=1}^t \frac{(n+1-r-t-s)}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}{}_{k_1 \cdots k_s i; l_1 \cdots l_s} H_{k_s}^{\alpha_b}, \end{aligned} \quad (4-109)$$

$$\begin{aligned} \sum_{l_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}{}_{k_1 \cdots k_s; l_1 \cdots l_s, l_s} &= \frac{1}{2} \left( \sum_{l_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{il_j}^{\alpha_{t+1}} \right. \\ &\quad \left. + \sum_{l_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}{}_{k_1 \cdots k_s i; l_1 \cdots l_s j} \bar{R}_{il_j}^{\alpha_b} \right) \\ &\quad - \sum_{l_s \alpha_{t+1}} \frac{r(n+1-r-t-s)}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}{}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{l_s}^{\alpha_{t+1}} \\ &\quad - \sum_{l_s} \sum_{b=1}^t \frac{(n+1-r-t-s)}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}{}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{l_s}^{\alpha_b}. \end{aligned} \quad (4-110)$$

**证明:** 对于一般 Newton 变换, 由定理 3.1 和命题 4.4

$$\begin{aligned} \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} h_{ij, k_s}^{\alpha_{t+1}} \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} h_{ij, k_s}^{\alpha_b} \end{aligned} \quad (4-111)$$

$$\begin{aligned} &= - \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} h_{ij, k_s}^{\alpha_{t+1}} \\ &- \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} h_{ij, k_s}^{\alpha_b} \end{aligned} \quad (4-112)$$

$$\begin{aligned} &= - \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} (h_{k_j, i}^{\alpha_{t+1}} + h_{ij, k_s}^{\alpha_{t+1}} - h_{k_j, i}^{\alpha_{t+1}}) \\ &- \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} (h_{k_j, i}^{\alpha_b} + h_{ij, k_s}^{\alpha_b} - h_{k_j, i}^{\alpha_b}) \end{aligned} \quad (4-113)$$

$$\begin{aligned} &= - \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} (h_{k_j, i}^{\alpha_{t+1}} + \bar{R}_{jk, i}^{\alpha_{t+1}}) \\ &- \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} (h_{k_j, i}^{\alpha_b} + \bar{R}_{jk, i}^{\alpha_b}) \end{aligned} \quad (4-114)$$

$$\begin{aligned} &= - \left( \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} h_{ij, k_s}^{\alpha_{t+1}} \right. \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} h_{ij, k_s}^{\alpha_b} \Big) \\ &+ \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \bar{R}_{jk, i}^{\alpha_{t+1}} \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \bar{R}_{jk, i}^{\alpha_b} \end{aligned} \quad (4-115)$$

$$\begin{aligned} &= - \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \\ &+ \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \bar{R}_{jk, i}^{\alpha_{t+1}} \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \bar{R}_{jk, i}^{\alpha_b}, \end{aligned} \quad (4-116)$$

$$\begin{aligned} \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= \frac{1}{2} \left( \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \bar{R}_{jk, i}^{\alpha_{t+1}} \right. \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \bar{R}_{jk, i}^{\alpha_b} \Big). \end{aligned} \quad (4-117)$$

对于迹零的 Newton 变换, 由定理 3.1 和命题 4.4 有:

$$\begin{aligned} \sum_{k_s} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, k_s}^{\alpha_{t+1}} \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, k_s}^{\alpha_b} \end{aligned} \quad (4-118)$$

$$\begin{aligned} &= - \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, k_s}^{\alpha_{t+1}} \\ &- \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, k_s}^{\alpha_b} \end{aligned} \quad (4-119)$$

$$\begin{aligned} &= - \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} (\hat{h}_{k_j, i}^{\alpha_{t+1}} + \hat{h}_{ij, k_s}^{\alpha_{t+1}} - \hat{h}_{k_j, i}^{\alpha_{t+1}}) \\ &- \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} (\hat{h}_{k_j, i}^{\alpha_b} + \hat{h}_{ij, k_s}^{\alpha_b} - \hat{h}_{k_j, i}^{\alpha_b}) \end{aligned} \quad (4-120)$$

$$\begin{aligned} &= - \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} \\ &(\hat{h}_{k_j, i}^{\alpha_{t+1}} + \bar{R}_{jk, i}^{\alpha_{t+1}} - \delta_{ij} H_{k_s}^{\alpha_{t+1}} + \delta_{jk_s} H_{i, i}^{\alpha_{t+1}}) \\ &- \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \\ &(\hat{h}_{k_j, i}^{\alpha_b} + \bar{R}_{jk_s, i}^{\alpha_b} - \delta_{ij} H_{k_s}^{\alpha_b} + \delta_{jk_s} H_{i, i}^{\alpha_b}) \end{aligned} \quad (4-121)$$

$$\begin{aligned} &= - \left( \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, k_s}^{\alpha_{t+1}} \right. \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, k_s}^{\alpha_b} \Big) \\ &+ \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} \bar{R}_{jk_s, i}^{\alpha_{t+1}} \\ &+ \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \bar{R}_{jk_s, i}^{\alpha_b} \\ &- \sum_{k_s \alpha_{t+1}} \frac{2r(n+1-r-t-s)}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{k_s}^{\alpha_{t+1}} \\ &- \sum_{k_s} \sum_{b=1}^t \frac{2(n+1-r-t-s)}{r+t} \widehat{T}_{(r, t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{k_s}^{\alpha_b} \end{aligned} \quad (4-122)$$

$$\begin{aligned} &= - \sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} \\ &+ \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} \widehat{T}_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} \bar{R}_{jk_s, i}^{\alpha_{t+1}} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} \bar{R}_{jk_s}^{\alpha_b} \\
& - \sum_{k_s \alpha_{t+1}} \frac{2r(n+1-r-t-s)}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{k_s}^{\alpha_{t+1}} \\
& - \sum_{k_s} \sum_{b=1}^t \frac{2(n+1-r-t-s)}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} H_{k_s}^{\alpha_b}
\end{aligned} \quad (4-123)$$

$$\begin{aligned}
\sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= \frac{1}{2} \left( \sum_{k_s} \sum_{ij} \sum_{\alpha_{t+1}} \frac{r}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} \bar{R}_{jk_s}^{\alpha_{t+1}} \right. \\
& + \sum_{k_s} \sum_{b=1}^t \sum_{ij} \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} \bar{R}_{jk_s}^{\alpha_b} \Big) \\
& - \sum_{k_s \alpha_{t+1}} \frac{r(n+1-r-t-s)}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{k_s}^{\alpha_{t+1}} \\
& - \sum_{k_s} \sum_{b=1}^t \frac{(n+1-r-t-s)}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} H_{k_s}^{\alpha_b}.
\end{aligned} \quad (4-124)$$

特别的, 对于  $N = R^{n+p}(c)$ , 有  $\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$ ,  $\bar{R}_{jk}^{\alpha} = 0$ .

**推论 4.3** (散度为零性质): 设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 有

$$\sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} = 0, \quad (4-125)$$

$$\begin{aligned}
\sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} &= - \sum_{k_s \alpha_{t+1}} \frac{r(n+1-r-t-s)}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} H_{k_s}^{\alpha_{t+1}} \\
& - \sum_{k_s} \sum_{b=1}^t \frac{(n+1-r-t-s)}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} H_{k_s}^{\alpha_b}.
\end{aligned} \quad (4-126)$$

**推论 4.4** (散度为零性质): 设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式中的具有平行平均曲率的子流形 ( $D\vec{H} = 0$ ), 有

$$\sum_{k_s} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} = 0, \quad (4-127)$$

$$\sum_{k_s} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, k_s} = 0. \quad (4-128)$$

**命题 4.6** (展开性质): 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 有

$$\begin{aligned}
T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} &= \delta_j^i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} - \sum_{p, \alpha_{t+1}} \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_t \alpha_{t+1}}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} h_{pj}^{\alpha_{t+1}} \\
& - \sum_{b=1}^t \sum_p \frac{1}{r+t} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots \hat{k}_s; l_1 \cdots l_s} h_{pj}^{\alpha_b} - \sum_{c=1}^s T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_s} \delta_j^{k_c} \quad (4-129)
\end{aligned}$$

$$\begin{aligned}
&= \delta_j^i T_{(r,t) \hat{k}_1 \dots \hat{k}_s i l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} - \sum_{p, \alpha_{t+1}} \frac{r}{(r+t)} T_{(r-2, t+1) \hat{k}_1 \dots \hat{k}_{p-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_t \alpha_{t+1}} h_{pi}^{\alpha_{t+1}} \\
&\quad - \sum_{b=1}^s \sum_p \frac{1}{r+t} T_{(r, t-1) \hat{k}_1 \dots \hat{k}_{p-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_b \dots \alpha_t} h_{pi}^{\alpha_b} - \sum_{c=1}^s T_{(r, t) \hat{k}_1 \dots \hat{k}_c \dots \hat{k}_{s-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \delta_{lc}^i,
\end{aligned} \quad (4-130)$$

$$\begin{aligned}
\widehat{T_{(r,t) \hat{k}_1 \dots \hat{k}_s i l_1 \dots l_s}^{\alpha_1 \dots \alpha_t}} &= \delta_j^i \widehat{T_{(r,t) \hat{k}_1 \dots \hat{k}_s i l_1 \dots l_s}^{\alpha_1 \dots \alpha_t}} - \sum_{p, \alpha_{t+1}} \frac{r}{(r+t)} \widehat{T_{(r-2, t+1) \hat{k}_1 \dots \hat{k}_{p-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_t \alpha_{t+1}}} \hat{h}_{pj}^{\alpha_{t+1}} \\
&\quad - \sum_{b=1}^s \sum_p \frac{1}{r+t} \widehat{T_{(r, t-1) \hat{k}_1 \dots \hat{k}_{p-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_b \dots \alpha_t}} \hat{h}_{pj}^{\alpha_b} - \sum_{c=1}^s \widehat{T_{(r, t) \hat{k}_1 \dots \hat{k}_c \dots \hat{k}_{s-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_t}} \delta_j^{k_c}
\end{aligned} \quad (4-131)$$

$$\begin{aligned}
&= \delta_j^i \widehat{T_{(r,t) \hat{k}_1 \dots \hat{k}_s i l_1 \dots l_s}^{\alpha_1 \dots \alpha_t}} - \sum_{p, \alpha_{t+1}} \frac{r}{(r+t)} \widehat{T_{(r-2, t+1) \hat{k}_1 \dots \hat{k}_{p-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_t \alpha_{t+1}}} \hat{h}_{pi}^{\alpha_{t+1}} \\
&\quad - \sum_{b=1}^s \sum_p \frac{1}{r+t} \widehat{T_{(r, t-1) \hat{k}_1 \dots \hat{k}_{p-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_b \dots \alpha_t}} \hat{h}_{pi}^{\alpha_b} - \sum_{c=1}^s \widehat{T_{(r, t) \hat{k}_1 \dots \hat{k}_c \dots \hat{k}_{s-1} l_1 \dots l_s}^{\alpha_1 \dots \alpha_t}} \delta_{lc}^i.
\end{aligned} \quad (4-132)$$

证明: 我们只需要广义 Kronecker 符号的展开性质。由引理 4.1 有

$$\begin{aligned}
\delta_{j_1 \dots j_r q_1 \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} &= \delta_j^i \delta_{j_1 \dots j_r q_1 \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} - \sum_{a=1}^r \delta_{j_1 \dots j_a q_1 \dots q_s l_1 \dots l_s}^{\hat{i}_1 \dots \hat{i}_a p_1 \dots p_s k_1 \dots k_s} \delta_j^{i_a} \\
&\quad - \sum_{b=1}^s \delta_{j_1 \dots j_r q_1 \dots q_b \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} \delta_j^{p_b} - \sum_{c=1}^s \delta_{j_1 \dots j_r q_1 \dots q_s l_1 \dots l_c \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} \delta_j^{k_c}
\end{aligned} \quad (4-133)$$

$$\begin{aligned}
T_{(r,t) \hat{k}_1 \dots \hat{k}_s i l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} &= \frac{1}{(r+t)!} \delta_{j_1 \dots j_r q_1 \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} \\
&\quad \times \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle (h_{p_1 q_1}^{\alpha_1} \dots h_{p_s q_s}^{\alpha_s})
\end{aligned} \quad (4-134)$$

$$\begin{aligned}
&= \frac{1}{(r+t)!} [\delta_j^i \delta_{j_1 \dots j_r q_1 \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} - \sum_{a=1}^r \delta_{j_1 \dots j_a q_1 \dots q_s l_1 \dots l_s}^{\hat{i}_1 \dots \hat{i}_a p_1 \dots p_s k_1 \dots k_s} \delta_j^{i_a} \\
&\quad - \sum_{b=1}^s \delta_{j_1 \dots j_r q_1 \dots q_b \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} \delta_j^{p_b} - \sum_{c=1}^s \delta_{j_1 \dots j_r q_1 \dots q_s l_1 \dots l_c \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} \delta_j^{k_c}] \\
&\quad \times \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle (h_{p_1 q_1}^{\alpha_1} \dots h_{p_s q_s}^{\alpha_s})
\end{aligned} \quad (4-135)$$

$$\begin{aligned}
&= \delta_j^i T_{(r,t) \hat{k}_1 \dots \hat{k}_s i l_1 \dots l_s}^{\alpha_1 \dots \alpha_t} \\
&\quad - \frac{r}{(r+t)(r-t-1)!} \delta_{j_1 \dots j_r-2 j_r-1 q_1 \dots q_s l_1 \dots l_s}^{i_1 \dots i_r-2 i_r-1 p_1 \dots p_s k_1 \dots k_s} \delta_j^{i_r} \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \\
&\quad \times \langle B_{i_{j_{r-3} j_r-3}}, B_{i_{j_{r-2} j_r-2}} \rangle \sum_{\alpha_{t+1}} h_{i_{j_r-1} i_{j_r}}^{\alpha_{t+1}} h_{i_{j_r}}^{\alpha_{t+1}} (h_{p_1 q_1}^{\alpha_1} \dots h_{p_s q_s}^{\alpha_s}) \\
&\quad - \frac{1}{r+t} \frac{1}{(r-t-1)!} \sum_{b=1}^s \delta_{j_1 \dots j_r q_1 \dots q_b \dots q_s l_1 \dots l_s}^{i_1 \dots i_r p_1 \dots p_s k_1 \dots k_s} \delta_j^{p_b} \\
&\quad \times \langle B_{i_{j_1}}, B_{i_{j_2}} \rangle \dots \langle B_{i_{j_{r-1}}}, B_{i_{j_r}} \rangle (h_{p_1 q_1}^{\alpha_1} \dots h_{p_s q_s}^{\alpha_s})
\end{aligned}$$

$$- \sum_{c=1}^s T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \delta_j^{k_c} \quad (4-136)$$

$$\begin{aligned} &= \delta_j^i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} - \sum_{p, \alpha_{t+1}} \frac{r}{(r+t)} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{t+1}}_{k_1 \cdots k_s; l_1 \cdots l_s} h_{pj}^{\alpha_{t+1}} \\ &\quad - \sum_{b=1}^t \sum_p \frac{1}{r+t} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} h_{pj}^{\alpha_b} - \sum_{c=1}^s T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \delta_j^{k_c}. \end{aligned} \quad (4-137)$$

特别的, 当  $r$  是偶数和  $t=0, s=1$  时我们有推论 4.5.

**推论 4.5** (参见文献[8]): 设  $x: M^n \rightarrow N^{n+p}$  是子流形,  $r$  是偶数时, 那么我们有

$$T_{(r)j}^i = \delta_j^i S_r - \sum_{p, \alpha} T_{(r-1)ip}^{\alpha} h_{pj}^{\alpha}, \quad (4-138)$$

$$\widehat{T}_{(r)\dot{y}} = \delta_{\dot{y}} \hat{S}_r - \sum_{p, \alpha} \widehat{T}_{(r-1)ip}^{\alpha} \hat{h}_{pj}^{\alpha}. \quad (4-139)$$

特别的, 当  $r$  是偶数和  $t=1, s=1$  时我们有推论 4.6.

**推论 4.6**: 设  $x: M^n \rightarrow N^{n+p}$  是子流形,  $r$  是偶数时, 那么我们有:

$$T_{(r,1)\dot{y}}^{\alpha} = \delta_{\dot{y}} S_{r+1}^{\alpha} - \frac{r}{r+1} \sum_{p, \beta} T_{(r-2, 2)}^{\alpha\beta}_{iip} h_{pj}^{\beta} - \frac{1}{r+1} \sum_p T_{(r)ip} h_{pj}^{\alpha}, \quad (4-140)$$

$$\widehat{T}_{(r,1)\dot{y}}^{\alpha} = \delta_{\dot{y}} \hat{S}_{r+1}^{\alpha} - \frac{r}{r+1} \sum_{p, \beta} \widehat{T}_{(r-2, 2)}^{\alpha\beta}_{iip} \hat{h}_{pj}^{\beta} - \frac{1}{r+1} \sum_p \widehat{T}_{(r)ip} \hat{h}_{pj}^{\alpha}. \quad (4-141)$$

**命题 4.7** (变分性质): 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 设  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 那么我们有

• 一般 Newton 变换

$$\begin{aligned} \frac{d}{dt} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} &= \sum_{\dot{y}} \left[ \sum_{\beta} \frac{r}{(r+t)} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\ &\quad \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \right]_{, \dot{y}} \\ &\quad - \sum_{\dot{y}} \left[ \frac{r}{(r+t)} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \right]_{, j} \\ &\quad - \sum_{\dot{y}} \left[ \frac{r}{(r+t)} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \right]_{, i} \\ &\quad + \sum_{\dot{y}} \left[ \frac{r}{(r+t)} T_{(r-2, t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r, t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \right] \\ &\quad + \sum_p T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^p - \sum_{c=1}^s \sum_i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_i^{k_c} \\ &\quad - \sum_{c=1}^s \sum_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_j^{l_c} - \sum_{b=1}^t \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_{\beta}^{\alpha_b} \\ &\quad + T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \langle \vec{S}_1, V \rangle - (r+t+1) \sum_{\beta} T_{(r,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \end{aligned}$$

$$\begin{aligned}
& - \sum_{c=1}^s \sum_{\beta} T_{r,t} \alpha_1 \cdots \alpha_t k_c \cdots k_{\beta} k_c i_1 \cdots i_c \cdots l_{\beta} h_{j|c}^{\beta} V^{\beta} - \sum_{i\beta\gamma} \left[ \frac{1}{(r+t)} T_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} \bar{R}_{ij\gamma}^{\beta} V^{\gamma} \right. \\
& \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} \bar{R}_{ij\gamma}^{\alpha b} V^{\gamma} \right]. \quad (4-142)
\end{aligned}$$

• 迹零 Newton 变换

$$\begin{aligned}
\frac{d}{dt} \widehat{T}_{(r,t)} \alpha_1 \cdots \alpha_t k_1 \cdots k_s i_1 \cdots l_s &= \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} + \right. \\
& \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \Big]_{,ij} \\
& - \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \right]_j \\
& - \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \right]_j \\
& + \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \right] \\
& - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \Big]_{,ij} \\
& + \sum_i \left[ \frac{2r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} \right. \\
& + \frac{2(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \Big]_{,i} \\
& - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} \right. \\
& + \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\alpha b} \Big] + \sum_p \widehat{T}_{(r,t)} \alpha_1 \cdots \alpha_t k_1 \cdots k_s i_1 \cdots l_s V^p \\
& - \sum_{c=1}^s \sum_i \widehat{T}_{(r,t)} \alpha_1 \cdots \alpha_t k_1 \cdots k_c \cdots k_{\beta} i_1 \cdots l_{\beta} L_i^{k_c} - \sum_{c=1}^s \sum_j \widehat{T}_{(r,t)} \alpha_1 \cdots \alpha_t k_1 \cdots k_c \cdots k_{\beta} k_c i_1 \cdots l_{\beta} L_j^{l_c} \\
& - \sum_{b=1}^t \sum_{\beta} \widehat{T}_{(r,t)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} L_{\beta}^{\alpha b} - [(r+t+1)] \sum_{\beta} \widehat{T}_{(r,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} V^{\beta} \\
& + \sum_{c=1}^s \sum_{\beta} \widehat{T}_{(r,t)} \alpha_1 \cdots \alpha_t k_1 \cdots k_c \cdots k_{\beta} k_c i_1 \cdots l_{\beta} h_{j|c}^{\beta} V^{\beta} + (r+t) \widehat{T}_{(r,t)} \alpha_1 \cdots \alpha_t k_1 \cdots k_s i_1 \cdots l_s < \vec{H}, V > \\
& + \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)} \alpha_1 \cdots \alpha_{\beta} k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} H^{\beta} \hat{h}_{pj}^{\gamma} V^{\gamma} \right. \\
& \left. + \sum_{b=1}^t \sum_{\beta} \frac{1}{(r+t)} \widehat{T}_{(r,t-1)} \alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t k_1 \cdots k_{\beta} i_1 \cdots l_{\beta} H^{\alpha b} \hat{h}_{pj}^{\gamma} V^{\gamma} \right]
\end{aligned}$$



$$\begin{aligned}
& - \left[ \frac{r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta} \hat{\sigma}_{\beta\gamma} V^\gamma \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} \hat{\sigma}_{ab\gamma} V^\gamma \Big] \\
& - \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta} \bar{R}_{ij\gamma}^\beta + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} \bar{R}_{ij\gamma}^{\alpha_b} \right] V^\gamma \\
& - \left[ \frac{r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta} \right]_{k_1 \cdots k_\beta; l_1 \cdots l_s} \bar{R}_{\beta\gamma}^\gamma \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} \bar{R}_{ab\gamma}^\gamma \Big] V^\gamma. \quad (4-143)
\end{aligned}$$

证明: 对于一般的 Newton 变换, 由定义和定理 3.3 以及引理 4.1

$$\begin{aligned}
\frac{\partial}{\partial t} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} &= \frac{r}{(r+t)} \frac{1}{(r-t-1)!} \delta_{j_1 \cdots j_{r-2j_r-1} p_1 \cdots p_{k_1} \cdots k_{j_r}}^{i_1 \cdots i_{r-2i_r-1} q_1 \cdots q_{l_1} \cdots l_{j_r}} < B_{ij_1}, B_{i_2 j_2} > \cdots \\
&\times < B_{i_{r-2j_{r-3}}, B_{i_{r-2j_{r-2}}} > \sum h_{i_{r-1} j_{r-1}}^{\alpha_{i+1}} \frac{\partial}{\partial t} (h_{i_{j_r}}^{\alpha_i}) (h_{i_{j_r}}^{\alpha_1}) (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_{q_1}}^{\alpha_t}) \\
&+ \frac{1}{r+t} \frac{1}{(r-t-1)!} \sum_{b=1}^t \delta_{j_1 \cdots j_{r-2j_r-1} q_b \cdots q_{k_1} \cdots k_{j_r}}^{i_1 \cdots i_{r-2i_r-1} p_b \cdots p_{l_1} \cdots l_{j_r}} < B_{ij_1}, B_{i_2 j_2} > \cdots \\
&< B_{i_{r-1} j_{r-1}}, B_{i_{j_r}} > (h_{p_1 q_1}^{\alpha_1} \cdots \frac{\partial}{\partial t} (h_{p_b q_b}^{\alpha_b}) \cdots h_{p_{q_t}}^{\alpha_t}) \quad (4-144)
\end{aligned}$$

$$\begin{aligned}
&= \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} \frac{\partial}{\partial t} (h_{ij}^\beta) \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} \frac{\partial}{\partial t} (h_{ij}^{\alpha_b}) \quad (4-145) \\
&= (T1) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} V_{ij}^\beta + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} V_{ij}^{\alpha_b} \right] \\
&+ (T2) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{ij,p}^\beta V^p + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{ij,p}^{\alpha_b} V^p \right] \\
&+ (T3) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{pj}^\beta L_i^j + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{pj}^{\alpha_b} L_i^j \right] \\
&+ (T4) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{pi}^\beta L_j^i + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{pi}^{\alpha_b} L_j^i \right] \\
&- (T5) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{ij}^\gamma L_\gamma^\beta + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{ij}^\gamma L_\gamma^{\alpha_b} \right] \\
&+ (T6) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{ip}^\beta h_{ij}^\gamma V^\gamma + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_s} h_{ip}^{\alpha_b} h_{ij}^\gamma V^\gamma \right] \\
&- (T7) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_s} \bar{R}_{ij\gamma}^\beta V^\gamma \right.
\end{aligned}$$

$$+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} \bar{R}_{ij\gamma}^{\alpha b} V^\gamma \Big] \quad (4-146)$$

对上式逐项计算。

对于(T1), 由命题4.5(散度性质), 有

$$(T1) = \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta_{,ij} \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b}_{,ij} \right] \quad (4-147)$$

$$= \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,ij} \\ - \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,i} \\ - \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,j} \\ + \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,i} \\ + \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,j} \\ + \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,i} \\ + \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^\beta \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^{\alpha b} \right]_{,j} \quad (4-148)$$

对于(T2), 由命题4.4(协变导数性质), 我们有

$$(T2) = \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} h^\beta_{ij,p} V^p \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} h^{\alpha b}_{ij,p} V^p \right] \quad (4-149)$$

$$= \sum_p T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} V^p, \quad (4-150)$$

对于(T3), 由命题4.6(展开性质), 我们有

$$(T3) = \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} h^\beta_{pj} \right. \\ \left. + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} h^{\alpha b}_{pj} \right] L^j_i \\ = [\delta^i_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j} - T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_{s,i} l_1 \cdots l_j}] L^j_i \quad (4-151)$$

$$- \sum_{c=1}^s T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; l_1 \cdots l_c} \delta_j^{k_c} L_i^i \quad (4-152)$$

$$= - \sum_{ij} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; l_1 \cdots l_c} L_i^j - \sum_{c=1}^s \sum_i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_c} L_i^{k_c}. \quad (4-153)$$

对于 (T4), 由命题 4.6 (展开性质), 我们有

$$\begin{aligned} (T4) &= \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_\beta} h_{pi}^\beta \right. \\ &\quad \left. + \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_j} h_{pi}^{\alpha_b} \right] L_j^i \end{aligned} \quad (4-154)$$

$$\begin{aligned} &= [\delta_j^i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_\beta} - T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_j} \\ &\quad - \sum_{c=1}^s T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_j} L_j^i] L_j^i \end{aligned} \quad (4-155)$$

$$= - \sum_{ij} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_j} L_j^i - \sum_{c=1}^s \sum_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_j} L_j^{l_c}. \quad (4-156)$$

对于 (T3) + (T4), 由  $L_j^i$  的反对称性, 我们有

$$\begin{aligned} (T3) + (T4) &= - \sum_{ij} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_j} L_i^i \\ &\quad - \sum_{c=1}^s \sum_i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_c} L_i^{k_c} - \sum_{ij} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_j} L_j^i \\ &\quad - \sum_{c=1}^s d_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_j} L_j^{l_c} \end{aligned} \quad (4-157)$$

$$\begin{aligned} &= - \sum_{c=1}^s \sum_i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_c} L_i^{k_c} \\ &\quad - \sum_{c=1}^s \sum_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_c; l_1 \cdots l_j} L_j^{l_c}. \end{aligned} \quad (4-158)$$

对于 (T5), 由定义和对称性及反对称性, 我们有

$$\begin{aligned} (T5) &= \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_\beta} h_{ij}^\gamma L_\gamma^\beta \right. \\ &\quad \left. + \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_\beta; l_1 \cdots l_j} h_{ij}^\gamma L_\gamma^{\alpha_b} \right] \end{aligned} \quad (4-159)$$

$$= r T_{(r-2,t+2)}^{\alpha_1 \cdots \alpha_\beta \gamma}_{k_1 \cdots k_\beta; l_1 \cdots l_\beta} L_\gamma^\beta + \sum_{b=1}^i T_{(r,t)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_\beta} L_\beta^{\alpha_b} \quad (4-160)$$

$$= \sum_{b=1}^i \sum_\beta T_{(r,t)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_\beta} L_\beta^{\alpha_b}. \quad (4-161)$$

对于 (T6), 由命题 4.6 (展开性质), 我们有

$$(T6) = \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_\beta; l_1 \cdots l_j} h_{pi}^\beta \right.$$

$$+ \sum \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_t, \beta_1 \cdots \beta_t} h_{\beta_1}^{\alpha_1} h_{\beta_2}^{\alpha_2} \cdots h_{\beta_t}^{\alpha_t} h_{ij}^{\gamma} V^{\gamma} \quad (4-162)$$

$$= [\delta_j^i T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} - T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} \delta_j^i] h_{ij}^{\gamma} V^{\gamma} \\ - \sum_{c=1}^s T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_s} \delta_j^{l_c} h_{ij}^{\gamma} V^{\gamma} \quad (4-163)$$

$$= T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} < \vec{S}_1, v > - T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} h_{ij}^{\beta} V^{\beta} \\ - \sum_{c=1}^s \sum_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_s} h_{j l_c}^{\gamma} h_{ij}^{\gamma} V^{\gamma} \quad (4-164)$$

$$= T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} < \vec{S}_1, V > - (r+t+1) \sum_{\beta} T_{(r,t+1)}^{\alpha_1 \cdots \alpha_{\beta}} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \\ - \sum_{c=1}^s \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_s} h_{j l_c}^{\beta} h_{ij}^{\beta} V^{\beta}. \quad (4-165)$$

对于(17), 保持不变。综上, 我们有

$$\frac{d}{dt} T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} = \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\ + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big]_{,ij} \\ - \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\ + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big]_{,j} \\ - \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\ + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big]_{,i} \\ + \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\ + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big] \\ + \sum_p T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s, p} V^p - \sum_{c=1}^s \sum_i T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_s} L_i^{k_c} \\ - \sum_{c=1}^s \sum_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_s} L_j^{l_c} \\ - \sum_{b=1}^t \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} L_{\beta}^{\alpha_b} - T_{(r,t)}^{\alpha_1 \cdots \alpha_t} h_{k_1 \cdots k_s; l_1 \cdots l_s} < \vec{S}_1, V > \\ - (r+t+1) \sum_{\beta} T_{(r,t+1)}^{\alpha_1 \cdots \alpha_{\beta}} h_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta}$$

$$\begin{aligned}
& - \sum_{c=1}^i \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_i}_{k_1 \cdots k_c \cdots k_{\beta} l_1 \cdots l_c \cdots l_j} \hat{h}_{j,c}^{\beta} V^{\beta} \\
& - \sum_{\substack{t \neq \gamma \\ t \neq \beta}} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \bar{R}_{ij\gamma}^{\beta} V^{\gamma} \right. \\
& \left. + \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \bar{R}_{ij\gamma}^{\alpha_b} V^{\gamma} \right], \quad (4-166)
\end{aligned}$$

对于迹零的 Newton 变换, 由定义和定理 3.3 以及引理 4.1 有

$$\begin{aligned}
\frac{\partial}{\partial t} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_i}_{k_1 \cdots k_s; l_1 \cdots l_s} &= \frac{r}{(r+t)} \frac{1}{(r-t-1)!} \delta_{j_1 \cdots j_r-2j_r-1p_1 \cdots p_{k_1} \cdots k_{j_r}}^{\alpha_1 \cdots \alpha_i} < \hat{B}_{i,j_1}, \hat{B}_{i,j_2} > \cdots \\
&\times < \hat{B}_{i_r-j_r-3}, \hat{B}_{i_r-2j_r-2} > \sum_{\alpha_{i+1}} \hat{h}_{i_r-1j_r-1}^{\alpha_{i+1}} \frac{\partial}{\partial t} (\hat{h}_{i_r}^{\alpha_{i+1}}) (h_{p_1 q_1}^{\alpha_1} \cdots h_{p_{\beta} q_{\beta}}^{\alpha_{\beta}}) \\
&+ \frac{1}{r+t} \frac{1}{(r-t-1)!} \sum_{b=1}^i \delta_{j_1 \cdots j_r-2j_r-1q_1 \cdots q_{\beta} l_1 \cdots l_{\beta}}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i} < \hat{B}_{i,j_1}, \hat{B}_{i,j_2} > \\
&\times < \hat{B}_{i_r-j_r-1}, \hat{B}_{i_r} > (\hat{h}_{p_1 q_1}^{\alpha_1} \cdots \frac{\partial}{\partial t} (\hat{h}_{p_{\beta} q_{\beta}}^{\alpha_{\beta}}) \cdots \hat{h}_{p_{\beta} q_{\beta}}^{\alpha_{\beta}}) \quad (4-167)
\end{aligned}$$

$$\begin{aligned}
&= \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \frac{\partial}{\partial t} (\hat{h}_{ij}^{\beta}) \\
&+ \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \frac{\partial}{\partial t} (\hat{h}_{ij}^{\alpha_b}) \quad (4-168)
\end{aligned}$$

$$\begin{aligned}
&= (T8) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} (V_{ij}^{\beta} - \frac{1}{n} \delta_{ij} \Delta V^{\beta}) \right. \\
&+ \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} (V_{ij}^{\alpha_b} - \frac{1}{n} \delta_{ij} \Delta V^{\alpha_b}) \left. \right] \\
&+ (T9) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \hat{h}_{ij,p}^{\beta} V^p \right. \\
&+ \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \hat{h}_{ij,p}^{\alpha_b} V^p \left. \right] \\
&+ (T10) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_p} \hat{h}_{ij}^{\beta} L_i^j \right. \\
&+ \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_p} \hat{h}_{ij}^{\alpha_b} L_i^j \left. \right] \\
&+ (T11) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \hat{h}_{li}^{\beta} L_j^i \right. \\
&+ \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \hat{h}_{li}^{\alpha_b} L_j^i \left. \right] \\
&- (T12) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \hat{h}_{ij}^{\beta} L_j^{\gamma} \right. \\
&+ \sum_{b=1}^i \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_i}_{k_1 \cdots k_{\beta} l_1 \cdots l_j} \hat{h}_{ij}^{\alpha_b} L_j^{\gamma} \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + (T13) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} \right. \\
& \times (\hat{h}_{ip}^{\beta} \hat{h}_{ij}^{\gamma} + \hat{h}_{pj}^{\beta} H^{\gamma} + H^{\beta} \hat{h}_{pj}^{\gamma} - \frac{1}{n} \delta_{pj} \hat{\sigma}_{\beta\gamma}) V^{\gamma} \\
& + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} \\
& \times (\hat{h}_{ip}^{\alpha_b} \hat{h}_{ij}^{\gamma} + \hat{h}_{pj}^{\alpha_b} H^{\gamma} + H^{\alpha_b} \hat{h}_{pj}^{\gamma} - \frac{1}{n} \delta_{pj} \hat{\sigma}_{\alpha_b \gamma}) V^{\gamma} \Big] \\
& - (T14) \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} (\bar{R}_{ij}^{\beta} + \frac{1}{n} \delta_{ij} \bar{R}_{\beta\gamma}^{\gamma}) V^{\gamma} \right. \\
& + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} (\bar{R}_{ij}^{\alpha_b} + \frac{1}{n} \delta_{ij} \bar{R}_{\alpha_b \gamma}^{\gamma}) V^{\gamma} \Big]
\end{aligned} \tag{4-169}$$

逐项计算上式中的 (T8) - (T14).

对于 (T8)

$$\begin{aligned}
(T8) &= \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} (V_{,ij}^{\beta} - \frac{1}{n} \delta_{ij} \Delta V^{\beta}) \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} (V_{,ij}^{\alpha_b} - \frac{1}{n} \delta_{ij} \Delta V^{\alpha_b}) \Big] \tag{4-170} \\
&= \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} V^{\beta} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_j}} V^{\alpha_b} \Big]_{,ij} \\
&- \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_{j,i}}} V^{\beta} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_{j,i}}} V^{\alpha_b} \Big]_j \\
&- \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_{ij}}} V^{\beta} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_{ij}}} V^{\alpha_b} \Big]_{,i} \\
&+ \sum_{ij} \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_{ij,i}}} V^{\beta} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_{ij,i}}} V^{\alpha_b} \Big] \\
&- \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_{\beta-1} l_1 \cdots l_s}} V^{\beta} \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big]_{,ii} \\
& + \sum_i \left[ \frac{2r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s, i} V^\beta \right. \\
& + \frac{2(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s, i} V^{\alpha_b} \Big]_{,i} \\
& - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s, ii} V^\beta \right. \\
& + \frac{(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s, ii} V^{\alpha_b} \Big]. \quad (4-171)
\end{aligned}$$

对于 (T9), 由命题 4.4 (协变导数性质), 有

$$\begin{aligned}
(T9) &= \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, p}^\beta V^p \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{ij, p}^{\alpha_b} V^p \Big] \quad (4-172)
\end{aligned}$$

$$= \sum_p T_{(r,t)}^{\widehat{\alpha_1 \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s, p} V^p. \quad (4-173)$$

对于 (T10), 由命题 4.6 (展开性质), 有

$$\begin{aligned}
(T10) &= \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{pj}^\beta \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{pj}^{\alpha_b} \Big] L_j^i \quad (4-174)
\end{aligned}$$

$$\begin{aligned}
&= \left[ \delta_j^i \widehat{T}_{r,t}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} - \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \right. \\
&- \sum_{c=1}^s \widehat{T}_{r,t}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \delta_j^{k_c} \Big] L_j^i \quad (4-175)
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{ij} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_j^i - \sum_{c=1}^s \sum_i \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_i^{k_c}. \quad (4-176)
\end{aligned}$$

对于 (T11), 由命题 4.6 (展开性质), 有

$$\begin{aligned}
(T11) &= \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{pi}^\beta \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}}_{k_1 \cdots k_s; l_1 \cdots l_s} \hat{h}_{pi}^{\alpha_b} \Big] L_j^i \quad (4-177)
\end{aligned}$$

$$\begin{aligned}
&= \left[ \delta_j^i \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} - \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \right. \\
&- \sum_{c=1}^s \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} \delta_j^{k_c} \Big] L_j^i \quad (4-178)
\end{aligned}$$

$$= - \sum_{ij} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} L_j^i - \sum_{c=1}^s \sum_j \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_p; i l_1 \cdots l_c \cdots l_j} L_j^{l_c}. \quad (4-179)$$

对于(T10) + (T11), 由  $L_j^i$  的反对称性, 有

$$\begin{aligned} (T10) + (T11) &= - \sum_{ij} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} L_j^i \\ &\quad - \sum_{c=1}^s \sum_i \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_p; i l_1 \cdots l_c \cdots l_j} L_i^{k_c} \\ &\quad - \sum_{ij} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} L_j^i \\ &\quad - \sum_{c=1}^s \sum_j \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_p; i l_1 \cdots l_c \cdots l_j} L_j^{l_c} \end{aligned} \quad (4-180)$$

$$\begin{aligned} &= - \sum_{c=1}^s \sum_i \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_p; i l_1 \cdots l_c \cdots l_j} L_i^{k_c} \\ &\quad - \sum_{c=1}^s \sum_j \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c; k_p; i l_1 \cdots l_c \cdots l_j} L_j^{l_c}. \end{aligned} \quad (4-181)$$

对于(T12), 由定义和对称性及反对称性, 有

$$\begin{aligned} (T12) &= \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_p \beta}_{k_1 \cdots k_p; i l_1 \cdots l_j} \hat{h}_{ij}^\gamma L_\gamma^\beta \right. \\ &\quad \left. + \sum_{b=1}^t \frac{1}{(r+1)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} \hat{h}_{ij}^\gamma L_\gamma^{\alpha_b} \right] \end{aligned} \quad (4-182)$$

$$= r \widehat{T}_{(r-2,t+2)}^{\alpha_1 \cdots \alpha_p \gamma}_{k_1 \cdots k_p; i l_1 \cdots l_j} L_\gamma^\beta + \sum_{b=1}^t \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} L_\beta^{\alpha_b} \quad (4-183)$$

$$= \sum_{b=1}^t \sum_\beta \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} L_\beta^{\alpha_b}. \quad (4-184)$$

对于(T13), 由命题4.6(展开性质), 我们有

$$\begin{aligned} (T13) &= \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_p \beta}_{k_1 \cdots k_p; i l_1 \cdots l_j} \right. \\ &\quad \times (\hat{h}_{ip}^\beta \hat{h}_{ij}^\gamma + \hat{h}_{pj}^\beta H^\gamma + H^\beta \hat{h}_{pj}^\gamma - \frac{1}{n} \delta_{pj} \hat{\sigma}_{\beta\gamma}) V^\gamma \\ &\quad + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} \\ &\quad \times (\hat{h}_{ip}^{\alpha_b} \hat{h}_{ij}^\gamma + \hat{h}_{pj}^{\alpha_b} H^\gamma + H^{\alpha_b} \hat{h}_{pj}^\gamma - \frac{1}{n} \delta_{pj} \hat{\sigma}_{\alpha_b\gamma}) V^\gamma \end{aligned} \quad (4-185)$$

$$\begin{aligned} &= \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_p \beta}_{k_1 \cdots k_p; i l_1 \cdots l_j} \hat{h}_{ip}^\beta \right. \\ &\quad \left. + \sum_{b=1}^t \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \alpha_b \cdots \alpha_t}_{k_1 \cdots k_p; i l_1 \cdots l_j} \hat{h}_{pi}^{\alpha_b} \right] \hat{h}_{ij}^\gamma V^\gamma \end{aligned}$$



$$\begin{aligned}
& + \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \hat{h}_{k_1 \cdots k_{sp}; l_1 \cdots l_{sj}}^{\beta} \hat{h}_{pj}^{\beta} \right. \\
& + \sum_{b=1}^t \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \hat{h}_{k_1 \cdots k_{sp}; l_1 \cdots l_{sj}}^{\alpha_b} \hat{h}_{pj}^{\alpha_b} \left. \right] H^\gamma V^\gamma \\
& + \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \hat{H}^\beta \hat{h}_{pj}^\gamma V^\gamma \right. \\
& + \sum_{b=1}^t \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \hat{H}^{\alpha_b} \hat{h}_{pj}^\gamma V^\gamma \left. \right] \\
& - \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \frac{1}{n} \delta_{pj} \hat{\sigma}_{\beta\gamma} V^\gamma \right. \\
& + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \frac{1}{n} \delta_{pj} \hat{\sigma}_{\alpha_b\gamma} V^\gamma \left. \right] \quad (4-186) \\
& = - \left[ (r+t+1) \right] \sum_{\beta} T_{(r,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} V^\beta \\
& + \sum_{c=1}^s \sum_{j\beta} T_{(r,t)}^{\widehat{\alpha_1 \cdots \alpha_t}} \hat{h}_{k_1 \cdots k_c; l_1 \cdots l_{sj}}^{\beta} \hat{h}_{j\beta}^\beta V^\beta \\
& + (r+t) T_{(r,t)}^{\widehat{\alpha_1 \cdots \alpha_t}} < \vec{H}, V > \\
& + \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \hat{H}^\beta \hat{h}_{pj}^\gamma V^\gamma \right. \\
& + \sum_{b=1}^t \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \hat{H}^{\alpha_b} \hat{h}_{pj}^\gamma V^\gamma \left. \right] \\
& - \left[ \frac{r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \hat{\sigma}_{\beta\gamma} V^\gamma \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \hat{\sigma}_{\alpha_b\gamma} V^\gamma \left. \right]. \quad (4-187)
\end{aligned}$$

对于(T14)

$$\begin{aligned}
(T14) & = \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} (\bar{R}_{ij\gamma}^\beta + \frac{1}{n} \delta_{ij} \bar{R}_{\beta\gamma}^\top) V^\gamma \right. \\
& + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} (\bar{R}_{ij\gamma}^{\alpha_b} + \frac{1}{n} \delta_{ij} \bar{R}_{\alpha_b\gamma}^\top) V^\gamma \left. \right] \quad (4-188)
\end{aligned}$$

$$\begin{aligned}
& = \left[ \frac{r}{(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \bar{R}_{ij\gamma}^\beta \right. \\
& + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \bar{R}_{ij\gamma}^{\alpha_b} \left. \right] V^\gamma \\
& + \left[ \frac{r(n+1-r-t-s)}{n(r+t)} T_{(r-2,t+1)}^{\widehat{\alpha_1 \cdots \alpha_\beta}} \bar{R}_{\beta\gamma}^\top \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} T_{(r,t-1)}^{\widehat{\alpha_1 \cdots \alpha_b \cdots \alpha_t}} \bar{R}_{\alpha_b\gamma}^\top \left. \right] V^\gamma. \quad (4-189)
\end{aligned}$$

综上, 我们有

$$\begin{aligned}
 \frac{d}{dt} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} = & \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_j} V^{\beta} \right. \\
 & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_j} V^{\alpha_b} \Big]_{,ij} \\
 & - \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_j, i} V^{\beta} \right. \\
 & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_j, i} V^{\alpha_b} \Big]_j \\
 & - \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_j, j} V^{\beta} \right. \\
 & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_j, j} V^{\alpha_b} \Big]_{,i} \\
 & + \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_j, ji} V^{\beta} \right. \\
 & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_j, ji} V^{\alpha_b} \Big] \\
 & - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\
 & + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big]_{,ii} \\
 & + \sum_i \left[ \frac{2r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s, i} V^{\beta} \right. \\
 & + \frac{2(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, j} V^{\alpha_b} \Big]_{,i} \\
 & - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s, ii} V^{\beta} \right. \\
 & + \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, ii} V^{\alpha_b} \Big] \\
 & + \sum_p \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, p} V^p - \sum_{c=1}^s \sum_i \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c \cdots k_s; l_1 \cdots \hat{l}_c \cdots l_s} L^{k_{ci}} \\
 & - \sum_{c=1}^s \sum_j \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c \cdots k_s; l_1 \cdots \hat{l}_c \cdots l_j} L_j^{l_c} \\
 & - \sum_{b=1}^t \sum_{\beta} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_{\beta}^{\alpha_b} \\
 & - \left[ (r+t+1) \sum_{\beta} \widehat{T}_{(r,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{c=1}^s \sum_{j\beta} \widehat{T}_{(r,t) \hat{k}_1 \cdots \hat{k}_c \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \alpha_t} \hat{h}_{jlc}^{\beta} V^{\beta}] \\
& + (r+t) \widehat{T}_{(r,t) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \alpha_t} \langle \vec{H}, V \rangle \\
& + \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \alpha_{\beta}} H^{\beta} \hat{h}_{pj}^{\gamma} V^{\gamma} \right. \\
& + \sum_{b=1}^t \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} H^{\alpha_b} \hat{h}_{pj}^{\gamma} V^{\gamma} ] \\
& - \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \alpha_{\beta}} \hat{\sigma}_{\beta\gamma} V^{\gamma} \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} \hat{\sigma}_{\alpha_b\gamma} V^{\gamma} ] \\
& - \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \alpha_{\beta}} \bar{R}_{ij\gamma}^{\beta} \right. \\
& + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} \bar{R}_{ij\gamma}^{\alpha_b} ] V^{\gamma} \\
& - \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \alpha_{\beta}} \bar{R}_{\beta\gamma}^{\gamma} \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} \bar{R}_{\alpha_b\gamma}^{\gamma} ] V^{\gamma}.
\end{aligned} \tag{4-190}$$

如果  $N^{n+p}$  是  $R^{n+p}(c)$ , 那么对于 (T7), (T14), 有

$$\begin{aligned}
(T7) &= \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \alpha_{\beta}} c \delta_{ij} V^{\beta} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} c \delta_{ij} V^{\alpha_b} ]
\end{aligned} \tag{4-191}$$

$$\begin{aligned}
&= c \left[ \sum_{i=1}^t \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \alpha_{\beta}} V^{\beta} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} V^{\alpha_b} ]
\end{aligned} \tag{4-192}$$

$$\begin{aligned}
&= \frac{r(n+1-r-t)c}{r+t} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \alpha_{\beta}} V^{\beta} \\
&+ \sum_{b=1}^t \frac{(n+1-r-t)c}{(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_s \hat{l}_1 \cdots \hat{l}_s}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} V^{\alpha_b}
\end{aligned} \tag{4-193}$$

$$\begin{aligned}
(T14) &= \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \alpha_{\beta}} (-c \delta_{ij} \delta_{\beta\gamma} + c \delta_{ij} \delta_{\beta\gamma}) V^{\gamma} \right. \\
&+ \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1) \hat{k}_1 \cdots \hat{k}_{s-1} \hat{l}_1 \cdots \hat{l}_j}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t} (-c \delta_{ij} \delta_{\gamma\alpha_b} + c \delta_{ij} \delta_{\alpha_b\gamma}) V^{\gamma} ] = 0.
\end{aligned} \tag{4-194}$$

**推论 4.7:** (变分性质)。设  $x: M^n \rightarrow R^{n+p}(c)$  是子流形, 设  $V = V^i e_i + V^\alpha e_\alpha$  是变向向量场, 则有

$$\begin{aligned} \frac{d}{dt} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} = & \left\{ \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j} V^{\beta} \right. \right. \\ & + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j} V^{\alpha_b} \Big]_{,ij} + \sum_p T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s, p} V^p \Big\} \\ & + \left\{ (n+1-r-t) c \left[ \sum_{\beta} \frac{r}{r+t} T_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \right. \\ & + \sum_{b=1}^t \frac{1}{(r+t)} T_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big] - (r+t+1) \sum_{\beta} T_{(r,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \Big\} \\ & + \{ T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} < \vec{S}_1, V > - \sum_{c=1}^s \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_c \cdots l_j} h_{j_c}^{\beta} V^{\beta} \} \\ & - \left\{ \sum_{c=1}^s \sum_i T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_c \cdots l_j} L_i^{k_c} + \sum_{c=1}^s \sum_j T_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_c \cdots k_s; l_1 \cdots l_c \cdots l_j} L_j^{l_c} \right. \\ & \left. + \sum_{b=1}^t \sum_{\beta} T_{(r,t)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} L_{\beta}^{\alpha_b} \right\}. \quad (4-195) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} = & \sum_{ij} \left[ \sum_{\beta} \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j} V^{\beta} \right. \\ & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j} V^{\alpha_b} \Big]_{,ij} \\ & - \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j, i} V^{\beta} \right. \\ & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j, i} V^{\alpha_b} \Big]_{,j} \\ & - \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j, j} V^{\beta} \right. \\ & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j, j} V^{\alpha_b} \Big]_{,i} \\ & + \sum_{ij} \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j, ji} V^{\beta} \right. \\ & + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_{\beta}; l_1 \cdots l_j, ji} V^{\alpha_b} \Big] \\ & - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\beta} \right. \\ & + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s; l_1 \cdots l_s} V^{\alpha_b} \Big]_{,ii} \\ & + \sum_i \left[ \frac{2r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_{\beta}}_{k_1 \cdots k_s; l_1 \cdots l_s, i} V^{\beta} \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{2(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s, i} V^{\alpha_b}]_{,i} \\
& - \sum_i \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_s i_1 \cdots i_s, ii} V^\beta \right. \\
& + \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s, ii} V^{\alpha_b} \left. \right] \\
& + \sum_p \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s, p} V^p - \sum_{c=1}^s \sum_l \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots \hat{k}_c \cdots k_s i_1 \cdots \hat{i}_c \cdots i_s} L_l^{k_c} \\
& - \sum_{c=1}^s \sum_j \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots \hat{k}_c \cdots k_s i_1 \cdots \hat{i}_c \cdots i_s} L_j^{i_c} - \sum_{b=1}^t \sum_\beta \widehat{T}_{(r,t)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s} L_\beta^{\alpha_b} \\
& - [(r+t+1) \sum_\beta \widehat{T}_{(r,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_s i_1 \cdots i_s} V^\beta \\
& + \sum_{c=1}^s \sum_{j\beta} \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots \hat{k}_c \cdots k_s i_1 \cdots \hat{i}_c \cdots i_s} \hat{h}_{j\beta}^\beta V^\beta] \\
& + (r+t) \widehat{T}_{(r,t)}^{\alpha_1 \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s} < \vec{H}, V > \\
& + \left[ \frac{r}{(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_s i_1 \cdots i_s} H^\beta \hat{h}_{pj}^\gamma V^\gamma \right. \\
& + \sum_{b=1}^t \frac{1}{(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s} H^{\alpha_b} \hat{h}_{pj}^\gamma V^\gamma \left. \right] \\
& - \left[ \frac{r(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r-2,t+1)}^{\alpha_1 \cdots \alpha_\beta}_{k_1 \cdots k_s i_1 \cdots i_s} \hat{\sigma}_{\beta\gamma} V^\gamma \right. \\
& + \sum_{b=1}^t \frac{(n+1-r-t-s)}{n(r+t)} \widehat{T}_{(r,t-1)}^{\alpha_1 \cdots \hat{\alpha}_b \cdots \alpha_t}_{k_1 \cdots k_s i_1 \cdots i_s} \hat{\sigma}_{\alpha\beta\gamma} V^\gamma \left. \right].
\end{aligned} \tag{4-196}$$

**推论 4.8:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 设  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 则有

•  $r$  是偶数

$$\begin{aligned}
\frac{d}{dt} S_r &= \sum_{ij} [T_{(r)}^{\alpha}_{ij} V^\alpha]_{,ij} - \sum_{ij} 2[T_{(r)}^{\alpha}_{ijj} V^\alpha]_{,i} \\
&+ \sum_{ij} [T_{(r)}^{\alpha}_{ij\hat{j}} V^\alpha] + \sum_p S_{r,p} V^p + S_r < \vec{S}_1, V > \\
&- (r+1) < \vec{S}_{r+1}, V > - \sum_{i\alpha\beta} T_{(r-1)}^{\alpha}_{ij\beta} V^\beta,
\end{aligned} \tag{4-197}$$

$$\begin{aligned}
\frac{d}{dt} \hat{S}_r &= [\widehat{T}_{(r-1)}^{\alpha}_{ij} V^\alpha]_{,ij} - 2[\widehat{T}_{(r-1)}^{\alpha}_{ij,i} V^\alpha]_{,j} + [\widehat{T}_{(r-1)}^{\alpha}_{ij\hat{j}} V^\alpha] \\
&- \frac{(n+1-r)}{n} [\hat{S}_{r-1}^{\alpha} V^\alpha]_{,\hat{u}} + \frac{2(n+1-r)}{n} [\hat{S}_{r-1,i}^{\alpha} V^\alpha]_{,i} - \frac{(n+1-r)}{n} [\hat{S}_{r-1,\hat{u}}^{\alpha} V^\alpha] \\
&+ \hat{S}_{r,p} V^p - (r+1) \hat{S}_{r+1}^{\alpha} V^\alpha + r \hat{S}_r < \vec{H}, V > + \widehat{T}_{(r-1)}^{\alpha}_{ij} H^\alpha \hat{h}_{ij}^\beta V^\beta
\end{aligned}$$

$$-\frac{(n+1-r)}{n}\hat{S}_{r-1}^{\alpha}\hat{\sigma}_{\alpha\phi}V^{\beta}-T_{(r-1)\hat{y}}^{\alpha}\hat{R}_{\hat{y}\beta}^{\alpha}V^{\beta}-\frac{(n+1-r)}{n}\hat{S}_{r-1}^{\alpha}\hat{R}_{\alpha\phi}^{\tau}V^{\beta}, \quad (4-198)$$

•  $r$  是奇数

$$\begin{aligned} \frac{d}{dt}S_r^{\alpha} &= \frac{d}{dt}T_{(r-1,1)\phi}^{\alpha} = \sum_{\hat{y}} \left[ \sum_{\beta} \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} V^{\alpha} \right]_{,\hat{y}} \\ &\quad - \sum_{\hat{y}} 2 \left[ \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} V^{\alpha} \right]_{,\hat{y}} \\ &\quad + \sum_{\hat{y}} \left[ \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} V^{\alpha} \right] \\ &\quad + \sum_p S_{r,p}^{\alpha} V^p - \sum_{\beta} S_r^{\beta} L_{\beta}^{\alpha} + S_r^{\alpha} \langle \vec{S}_1, V \rangle - (r+1) \sum_{\beta} T_{(r-1,2)\phi}^{\alpha\beta} V^{\beta} \\ &\quad - \sum_{\hat{y}\hat{\gamma}} \left[ \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} \hat{R}_{\hat{y}\hat{\gamma}}^{\beta} V^{\gamma} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} \hat{R}_{\hat{y}\hat{\gamma}}^{\alpha} V^{\gamma} \right], \quad (4-199) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\hat{S}_r^{\alpha} &= \left[ \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} V^{\alpha} \right]_{,\hat{y}} \\ &\quad - 2 \left[ \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} V^{\alpha} \right]_{,\hat{y}} \\ &\quad + \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} V^{\alpha} \\ &\quad - \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)\phi}^{\alpha\beta} V^{\beta} + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^{\alpha} V^{\alpha} \right]_{,\hat{y}} \\ &\quad + 2 \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)\phi,\hat{y}}^{\alpha\beta} V^{\beta} + \frac{(n+1-r)}{nr} \hat{S}_{r-1,\hat{y}}^{\alpha} V^{\alpha} \right]_{,\hat{y}} \\ &\quad - \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)\phi,\hat{y}}^{\alpha\beta} V^{\beta} + \frac{(n+1-r)}{nr} \hat{S}_{r-1,\hat{y}}^{\alpha} V^{\alpha} \right] \\ &\quad + \hat{S}_{r,p}^{\alpha} V^p - \hat{S}_r^{\beta} L_{\beta}^{\alpha} = (r+1) T_{(r-1,2)\phi}^{\alpha\beta} V^{\beta} + r \hat{S}_r^{\alpha} \langle \vec{H}, V \rangle \\ &\quad + \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} H_{\hat{y}}^{\beta} \hat{h}_{\hat{y}}^{\gamma} V^{\gamma} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} H_{\hat{y}}^{\alpha} \hat{h}_{\hat{y}}^{\gamma} V^{\gamma} \\ &\quad - \left[ \frac{(r-1)(n+1-r)}{nr} T_{(r-3,2)\phi}^{\alpha\beta} \hat{\sigma}_{\beta\gamma} V^{\gamma} + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^{\alpha} \hat{\sigma}_{\alpha\gamma} V^{\gamma} \right] \\ &\quad - \left[ \frac{r-1}{r} T_{(r-3,2)\hat{y}}^{\alpha\beta} \right]_{,\hat{y}} \hat{R}_{\hat{y}\hat{\gamma}}^{\beta} V^{\gamma} + \frac{1}{r} T_{(r-1)\hat{y}}^{\alpha} \hat{R}_{\hat{y}\hat{\gamma}}^{\alpha} V^{\gamma} \\ &\quad - \left[ \frac{(r-1)(n+1-r)}{nr} T_{(r-3,2)\phi}^{\alpha\beta} \right]_{,\hat{y}} \hat{R}_{\beta\gamma}^{\alpha} V^{\gamma} + \frac{(n+1-r)}{nr} \hat{S}_{(r-1)}^{\alpha} \hat{R}_{\alpha\gamma}^{\tau} V^{\gamma}. \quad (4-200) \end{aligned}$$

**推论 4.9:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 设  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 则有

•  $r$  为偶数

$$\frac{d}{dt} \int_M \dot{S}_r dv = \int_M \sum_{ij} [T_{(r-1)ij}^{\alpha} V^{\alpha}] - (r+1) \langle \vec{S}_{r+1}, V \rangle - \sum_{ij\alpha\beta} T_{(r-1)ij}^{\alpha} \bar{R}_{ij\beta}^{\alpha} V^{\beta}, \quad (4-201)$$

$$\begin{aligned} \frac{d}{dt} \int_M \dot{S}_r dv &= \int_M [\widehat{T_{(r-1)ij}^{\alpha}} V^{\alpha}] - \frac{(n+1-r)}{n} [\hat{S}_{r-1,ii}^{\alpha} V^{\alpha}] \\ &\quad - (r+1) \hat{S}_{r+1}^{\alpha} V^{\alpha} - (n-r) \hat{S}_r \langle \vec{H}, V \rangle + \widehat{T_{(r-1)ij}^{\alpha}} H^{\alpha} \hat{h}_{ij}^{\beta} V^{\beta} \\ &\quad - \frac{(n+1-r)}{n} \hat{S}_{r-1}^{\alpha} \hat{\sigma}_{\alpha\beta} V^{\beta} - \widehat{T_{(r-1)ij}^{\alpha}} \bar{R}_{ij\beta}^{\alpha} V^{\beta} - \frac{(n+1-r)}{n} \hat{S}_{r-1}^{\alpha} \bar{R}_{\alpha\beta}^{\alpha} V^{\beta}. \end{aligned} \quad (4-202)$$

•  $r$  为奇数

$$\begin{aligned} \frac{d}{dt} \int_M |\vec{S}_r|^2 dv &= \int_M 2S_{r,ji}^{\alpha} [\sum_{\beta} \frac{r-1}{r} T_{(r-3,2)ij}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)ij}^{\alpha} V^{\alpha}] \\ &\quad + 4S_{r,j}^{\alpha} [\frac{r-1}{r} T_{(r-3,2)ij,i}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)ij,i}^{\alpha} V^{\alpha}] + |\vec{S}_r|^2 \langle \vec{S}_1, V \rangle \\ &\quad + \sum_{ij} 2S_r^{\alpha} [\frac{r-1}{r} T_{(r-3,2)ij,ji}^{\alpha\beta} V^{\beta} + \frac{1}{r} T_{(r-1)ij,ji}^{\alpha} V^{\alpha}] \\ &\quad - 2(r+1) \sum_{\beta} S_r^{\alpha} T_{(r-1,2)\phi}^{\alpha\beta} V^{\beta} \\ &\quad - \sum_{ij\beta\gamma} 2S_r^{\alpha} [\frac{r-1}{r} T_{(r-3,2)ij}^{\alpha\beta} \bar{R}_{ij\gamma}^{\beta} V^{\gamma} + \frac{1}{r} T_{(r-1)ij}^{\alpha} \bar{R}_{ij}^{\alpha} V^{\gamma}] dv, \end{aligned} \quad (4-203)$$

$$\begin{aligned} \frac{d}{dt} \int_M |\vec{S}_r|^2 dv &= \int_M 2\hat{S}_{r,ji}^{\alpha} [\frac{r-1}{r} \widehat{T_{(r-3,2)ij}^{\alpha\beta}} V^{\beta} + \frac{1}{r} \widehat{T_{(r-1)ij}^{\alpha}} V^{\alpha}] \\ &\quad + 4\hat{S}_{r,j}^{\alpha} [\frac{r-1}{r} \widehat{T_{(r-3,2)ij,i}^{\alpha\beta}} V^{\beta} + \frac{1}{r} \widehat{T_{(r-1)ij,i}^{\alpha}} V^{\alpha}] \\ &\quad + 2\hat{S}_r^{\alpha} [\frac{r-1}{r} \widehat{T_{(r-3,2)ij,ji}^{\alpha\beta}} V^{\beta} + \frac{1}{r} \widehat{T_{(r-1)ij,ji}^{\alpha}} V^{\alpha}] \\ &\quad - 2\hat{S}_{r,ii}^{\alpha} [\frac{(r-1)(n+1-r)}{(nr)} \widehat{T_{(r-3,2)\phi}^{\alpha\beta}} V^{\beta} + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^{\alpha} V^{\alpha}] \\ &\quad - 4\hat{S}_{r,i}^{\alpha} [\frac{(r-1)(n+1-r)}{(nr)} \widehat{T_{(r-3,2)\phi,i}^{\alpha\beta}} V^{\beta} + \frac{(n+1-r)}{nr} \hat{S}_{r-1,i}^{\alpha} V^{\alpha}] \\ &\quad - 2\hat{S}_r^{\alpha} [\frac{(r-1)(n+1-r)}{(nr)} \widehat{T_{(r-3,2)\phi,ii}^{\alpha\beta}} V^{\beta} + \frac{(n+1-r)}{nr} \hat{S}_{r-1,ii}^{\alpha} V^{\alpha}] \\ &\quad - 2(r+1) \hat{S}_r^{\alpha} \widehat{T_{(r-1,2)\phi}^{\alpha\beta}} V^{\beta} - (n-2r) |\vec{S}_r|^2 \langle \vec{H}, V \rangle \\ &\quad + \frac{r-1}{r} 2\hat{S}_r^{\alpha} \widehat{T_{(r-3,2)ij}^{\alpha\beta}} H^{\beta} \hat{h}_{ij}^{\gamma} V^{\gamma} + \frac{1}{r} 2\hat{S}_r^{\alpha} \widehat{T_{(r-1)ij}^{\alpha}} H^{\alpha} \hat{h}_{ij}^{\gamma} V^{\gamma} \\ &\quad - 2\hat{S} [\frac{(r-1)(n+1-r)}{nr} \widehat{T_{(r-3,2)\phi}^{\alpha\beta}} \hat{\sigma}_{\beta\gamma} V^{\gamma} + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^{\alpha} \hat{\sigma}_{\alpha\gamma} V^{\gamma}] \end{aligned}$$

$$\begin{aligned}
& -2 \hat{S}_r^\alpha \left[ \frac{r-1}{r} T_{(r-3,2)}^\alpha \right]_{ij}^{\alpha\beta} \bar{R}_{\beta\gamma}^\beta V^\gamma + \frac{1}{r} T_{(r-1)\beta}^\alpha \bar{R}_{\beta\gamma}^\beta V^\gamma \\
& -2 \hat{S}_r^\alpha \left[ \frac{(r-1)(n+1-r)}{nr} T_{(r-3,2)\phi}^\alpha \right]_{\phi}^{\alpha\beta} \bar{R}_{\beta\gamma}^\beta V^\gamma \\
& + \frac{(n+1-r)}{nr} \hat{S}_{(r-1)}^\alpha \bar{R}_{\alpha\gamma}^\beta V^\gamma \big] dv. \tag{4-204}
\end{aligned}$$

如果  $N^{n+p} = R^{n+p}(c)$ , 那么  $\sum_{\beta} \bar{R}_{ij\beta}^\alpha V^\beta = -c \delta_{ij} V^\alpha$ .

**推论 4.10:** 设  $x: M^n \rightarrow R^{n+p}(c)$  是子流形, 设  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 则有

•  $r$  是偶数, 参见文献[7]

$$\begin{aligned}
\frac{d}{dt} S_r &= \sum_{ij} [T_{(r)}^\alpha V^\alpha]_{,ij} + \sum_p S_{r,p} V^p + S_r \langle \vec{S}_1, V \rangle \\
&- (r+1) \langle \vec{S}_{r+1}, V \rangle + c(n-r+1) \langle \vec{S}_{r-1}, V \rangle, \tag{4-205}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \hat{S}_r &= [T_{(r-1)}^\alpha V^\alpha]_{,ij} - 2 [T_{(r-1)}^\alpha V^\alpha]_{,j} + [T_{(r-1)}^\alpha V^\alpha]_{,i} \\
&- \frac{(n+1-r)}{n} [\hat{S}_{r-1}^\alpha V^\alpha]_{,ii} + \frac{2(n+1-r)}{n} [\hat{S}_{r-1,i}^\alpha V^\alpha]_{,i} - \frac{(n+1-r)}{n} [\hat{S}_{r-1,ii}^\alpha V^\alpha] \\
&+ \hat{S}_{r,p} V^p - (r+1) \hat{S}_{r+1}^\alpha V^\alpha + r \hat{S}_r \langle \vec{H}, V \rangle + T_{(r-1)\beta}^\alpha \hat{H}_{ij}^\beta V^\beta \\
&- \frac{(n+1-r)}{n} \hat{S}_{r-1}^\alpha \hat{\sigma}_{\alpha\beta} V^\beta. \tag{4-206}
\end{aligned}$$

•  $r$  是奇数

$$\begin{aligned}
\frac{d}{dt} S_r^\alpha &= \frac{d}{dt} T_{(r-1,1)\phi}^\alpha = \sum_{ij} \left[ \sum_{\beta} \frac{r-1}{r} T_{(r-3,2)}^{\alpha\beta} V^\beta + \frac{1}{r} T_{(r-1)\beta}^\alpha V^\beta \right]_{,ij} \\
&+ \sum_p S_{r,p}^\alpha V^p - \sum_{\beta} S_r^\beta L_{\beta}^\alpha + S_r^\alpha \langle \vec{S}_1, V \rangle - (r+1) \sum_{\beta} T_{(r-1,2)\phi}^{\alpha\beta} V^\beta \\
&+ \frac{c(r-1)(n-r+1)}{r} T_{(r-3,2)\phi}^{\alpha\beta} V^\beta + \frac{c(n-r+1)}{r} S_{r-1}^\alpha V^\alpha, \tag{4-207}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \hat{S}_r^\alpha &= \left[ \frac{r-1}{r} T_{(r-3,2)i}^\alpha V^\beta + \frac{1}{r} T_{(r-1)\beta}^\alpha V^\beta \right]_{,ij} \\
&- 2 \left[ \frac{r-1}{r} T_{(r-3,2)}^\alpha \right]_{ij,i} V^\beta + \frac{1}{r} T_{(r-1)\beta,i}^\alpha V^\beta \big]_{,j} \\
&+ \frac{r-1}{r} T_{(r-3,2)\beta}^\alpha V^\beta + \frac{1}{r} T_{(r-1)\beta,i}^\alpha V^\beta \\
&- \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)\phi}^\alpha \right]_{\phi}^{\alpha\beta} V^\beta + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^\alpha V^\alpha \big]_{,ii} \\
&+ 2 \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)\phi,i}^\alpha V^\beta + \frac{(n+1-r)}{nr} \hat{S}_{r-1,i}^\alpha V^\alpha \right]_{,i}
\end{aligned}$$



$$\begin{aligned}
& - \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{\phi,\dot{u}}{\phantom{a}}} V^\beta + \frac{(n+1-r)}{nr} \dot{S}_{r-1,\dot{u}} V^\alpha \right] \\
& + \dot{S}_{r,p}^\alpha V^p - \dot{S}_r^\beta L_\beta^\alpha - (r+1) T_{(r-1,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{\phi}{\phantom{a}}} V^\beta + r \dot{S}_r^\alpha \langle \vec{H}, V \rangle \\
& + \frac{r-1}{r} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{i,j}{\phantom{a}}} H^\beta \dot{h}_{ij}^\gamma V^\gamma + \frac{1}{r} T_{(r-1)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{ij}{\phantom{a}}} H^\alpha \dot{h}_{ij}^\gamma V^\gamma \\
& - \left[ \frac{(r-1)(n+1-r)}{nr} T_{(r-3,2)}^{\widehat{\phantom{a}}} \right] \overset{\alpha\beta}{\underset{\phi}{\phantom{a}}} \dot{\sigma}_{\beta\gamma} V^\gamma \\
& + \frac{(n+1-r)}{nr} \dot{S}_{r-1}^\alpha \dot{\sigma}_{\alpha\gamma} V^\gamma, \tag{4-208}
\end{aligned}$$

**推论 4.11:** 设  $x: M^n \rightarrow R^{n+p}(c)$  具有平行平均曲率的子流形 ( $D\vec{H}=0$ ), 设  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么我们有

•  $r$  是偶数

$$\begin{aligned}
\frac{d}{dt} S_r &= \sum_{ij} [T_{(r)}^{\alpha} V^\alpha]_{,ij} + \sum_p S_{r,p} V^p + S_r \langle \vec{S}_1, V \rangle \\
& - (r+1) \langle \vec{S}_{r+1}, V \rangle + c(n-r+1) \langle \vec{S}_{r-1}, V \rangle, \tag{4-209}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \dot{S}_r &= [T_{(r-1)}^{\alpha} V^\alpha]_{,ij} - \frac{(n+1-r)}{n} [\dot{S}_{r-1}^\alpha V^\alpha]_{,ii} \\
& + \frac{2(n+1-r)}{n} [\dot{S}_{r-1,i}^\alpha V^\alpha]_{,i} - \frac{(n+1-r)}{n} [\dot{S}_{r-1,\dot{u}}^\alpha V^\alpha] \\
& + \dot{S}_{r,p}^\alpha V^p - (r+1) \dot{S}_{r+1}^\alpha V^\alpha + r \dot{S}_r^\alpha \langle \vec{H}, V \rangle + T_{(r-1)}^{\widehat{\phantom{a}}} \overset{\alpha H^\alpha}{\underset{ij}{\phantom{a}}} \dot{h}_{ij}^\beta V^\beta \\
& - \frac{(n+1-r)}{n} \dot{S}_{r-1}^\alpha \dot{\sigma}_{\alpha\beta} V^\beta. \tag{4-210}
\end{aligned}$$

•  $r$  是奇数

$$\begin{aligned}
\frac{d}{dt} S_r^\alpha &= \frac{d}{dt} T_{(r-1,1)}^{\alpha} = \sum_{ij} \left[ \sum_{\beta} \frac{r-1}{r} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{i,j}{\phantom{a}}} V^\beta + \frac{1}{r} T_{(r-1)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{ij}{\phantom{a}}} V^\beta \right]_{,ij} \\
& + \sum_p S_{r,p}^\alpha V^p - \sum_{\beta} S_r^\beta L_\beta^\alpha + S_r^\alpha \langle \vec{S}_1, V \rangle - (r+1) \sum_{\beta} T_{(r-1,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{\phi}{\phantom{a}}} V^\beta \\
& + \frac{c(r-1)(n-r+1)}{r} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{\phi}{\phantom{a}}} V^\beta + \frac{c(n-r+1)}{r} S_{r-1}^\alpha V^\alpha, \tag{4-211}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \dot{S}_r^\alpha &= \left[ \frac{r-1}{r} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{ij}{\phantom{a}}} V^\beta + \frac{1}{r} T_{(r-1)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{ij}{\phantom{a}}} V^\beta \right]_{,ij} \\
& - \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{\phi}{\phantom{a}}} V^\beta + \frac{(n+1-r)}{nr} \dot{S}_{r-1}^\alpha V^\alpha \right]_{,ii} \\
& + 2 \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)}^{\widehat{\phantom{a}}} \overset{\alpha\beta}{\underset{\phi,i}{\phantom{a}}} V^\beta + \frac{(n+1-r)}{nr} \dot{S}_{r-1,i}^\alpha V^\alpha \right]_{,i} \\
& - \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)}^{\widehat{\phantom{a}}} \right] \overset{\alpha\beta}{\underset{\phi,\dot{u}}{\phantom{a}}} V^\beta + \frac{(n+1-r)}{nr} \dot{S}_{r-1,\dot{u}}^\alpha V^\alpha
\end{aligned}$$

$$\begin{aligned}
& + \hat{S}_{r,p}^\alpha V^p - \hat{S}_\gamma^\beta L_\beta^\alpha - (r+1) T_{(r-1,2)}^\alpha \hat{\phi}^{\alpha\beta} V^\beta + r \hat{S}_r^\alpha \langle \vec{H}, V \rangle \\
& + \frac{r-1}{r} T_{(r-3,2)}^\alpha \hat{i}_{ij}^{\alpha\beta} H^\beta \hat{h}_{ij}^\gamma V^\gamma + \frac{1}{r} T_{(r-1)}^\alpha H^\alpha \hat{h}_{ij}^\gamma V^\gamma \\
& - \left[ \frac{(r-1)(n+1-r)}{nr} \right] T_{(r-3,2)}^\alpha \hat{\phi}^{\alpha\beta} \hat{\sigma}_{\beta\gamma} V^\gamma \\
& + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^\alpha \hat{\sigma}_{\alpha\gamma} V^\gamma ].
\end{aligned} \quad (4-212)$$

**推论 4.12:** 设  $x: M^n \rightarrow R^{n+p}(c)$  是紧致无边子流形, 设  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场,  $r$  是偶数, 那么我们有

•  $r$  是偶数, 参见文献<sup>[7]</sup>

$$\begin{aligned}
\frac{d}{dt} \int_M S_i dv_i &= \int_M - (r+1) \langle \vec{S}_{r+1}, V \rangle \\
&+ c(n-r+1) \langle \vec{S}_{r-1}, V \rangle dv_i;
\end{aligned} \quad (4-213)$$

$$\begin{aligned}
\frac{d}{dt} \int_M \hat{S}_r dv_i &= \int_M (T_{(r-1)}^\alpha \hat{i}_{ij,ji} V^\alpha) - \frac{(n+1-r)}{n} [\hat{S}_{r-1,i}^\alpha V^\alpha] \\
&- (r+1) \hat{S}_{r+1}^\alpha V^\alpha - (n-r) \hat{S}_r^\alpha \langle \vec{H}, V \rangle + T_{(r-1)}^\alpha \hat{i}_{ij}^{\alpha\beta} H^\beta V^\beta \\
&- \frac{(n+1-r)}{n} \hat{S}_{r-1}^\alpha \hat{\sigma}_{\alpha\beta} V^\beta.
\end{aligned} \quad (4-214)$$

•  $r$  是奇数

$$\begin{aligned}
\frac{d}{dt} \int_M |\vec{S}_r|^2 dv &= \int_M 2 S_{r,ji}^\alpha \left[ \sum_\beta \frac{r-1}{r} T_{(r-3,2)}^\alpha \hat{i}_{ij}^{\alpha\beta} V^\beta + \frac{1}{r} T_{(r-1)}^\alpha V^\alpha \right] \\
&+ |\vec{S}_r|^2 \langle \vec{S}_1, V \rangle - 2(r+1) \sum_\beta S_r^\alpha T_{(r-1,2)}^\alpha \hat{\phi}^{\alpha\beta} V^\beta \\
&+ 2c S_r^\alpha \left[ \frac{(r-1)(n-r+1)}{r} T_{(r-3,2)}^\alpha \hat{\phi}^{\alpha\beta} V^\beta \right. \\
&\left. + \frac{(n+1-r)}{r} S_{(r-1)}^\alpha V^\alpha \right],
\end{aligned} \quad (4-215)$$

$$\begin{aligned}
\frac{d}{dt} \int_M |\vec{S}|^2 dv &= \int_M 2 \hat{S}_{r,ji}^\alpha \left[ \frac{r-1}{r} T_{(r-3,2)}^\alpha \hat{i}_{ij}^{\alpha\beta} V^\beta + \frac{1}{r} T_{(r-1)}^\alpha V^\alpha \right] \\
&+ 4 \hat{S}_{r,j}^\alpha \left[ \frac{r-1}{r} T_{(r-3,2)}^\alpha \hat{i}_{ij,i}^{\alpha\beta} V^\beta + \frac{1}{r} T_{(r-1)}^\alpha \hat{i}_{ij,i} V^\alpha \right] \\
&+ 2 \hat{S}_r^\alpha \left[ \frac{r-1}{r} T_{(r-3,2)}^\alpha \hat{i}_{ij,ji}^{\alpha\beta} V^\beta + \frac{1}{r} T_{(r-1)}^\alpha \hat{i}_{ij,ji} V^\alpha \right] \\
&- 2 \hat{S}_{r,ii}^\alpha \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)}^\alpha \hat{\phi}^{\alpha\beta} V^\beta + \frac{(n+1-r)}{nr} \hat{S}_{r-1}^\alpha V^\alpha \right] \\
&- 4 \hat{S}_{r,i}^\alpha \left[ \frac{(r-1)(n+1-r)}{(nr)} T_{(r-3,2)}^\alpha \hat{\phi}_{\phi,i}^{\alpha\beta} V^\beta + \frac{(n+1-r)}{nr} \hat{S}_{r-1,i}^\alpha V^\alpha \right]
\end{aligned}$$

$$\begin{aligned}
& -2 \hat{S}_r^\alpha \left[ \frac{(r-1)(n+1-r)}{(nr)} \widehat{T_{(r-3,2)}^\alpha} \right]_{\phi, \hat{u}}^{\alpha\beta} V^\beta + \frac{(n+1-r)}{nr} \hat{S}_{r-1, \hat{u}} V^\alpha \\
& + -2(r+1) \hat{S}_r^\alpha \widehat{T_{(r-1,2)}^\alpha} V^\beta - (n-2r) |\hat{S}|^2 < \vec{H}, V > \\
& + \frac{r-1}{r} 2 \hat{S}_r^\alpha \widehat{T_{(r-3,2)}^\alpha} H_{ij}^\beta \hat{h}_{ij}^\gamma V^\gamma + \frac{1}{2} \hat{S}_r^\alpha \widehat{T_{(r-1)}^\alpha} H^\alpha \hat{h}_{ij}^\gamma V^\gamma \\
& - 2 \hat{S}_r^\alpha \left[ \frac{(r-1)(n+1-r)}{nr} \widehat{T_{(r-3,2)}^\alpha} \right]_{\phi}^{\alpha\beta} \hat{\sigma}_{\beta\gamma} V^\gamma \\
& + \frac{(n+1-r)}{nr} \hat{S}_{r-1} \hat{\sigma}_{\alpha\gamma} V^\gamma.
\end{aligned} \tag{4-216}$$

对于超曲面的情形, 我们有

**推论 4.13:** 变分性质. 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 设  $V = V^i e_i + fN$  是变分向量场, 那么我们有

$$\begin{aligned}
\frac{d}{dt} T_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} &= \sum_{ij} [T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} f]_{,ij} - \sum_{ij} [T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} f]_{,j} - \sum_{ij} [T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} f]_{,i} \\
&+ \sum_{ij} [T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} f]_{,ij} + \sum_p T_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s, p} V^p \\
&- \sum_{c=1}^s \sum_i T_{(r)}^{k_1 \cdots \hat{k}_c \cdots k_s}_{l_1 \cdots l_c \cdots l_s} L_i^{k_c} + \sum_{c=1}^s \sum_i T_{(r)}^{k_1 \cdots \hat{k}_c \cdots k_s}_{l_1 \cdots l_c \cdots l_s} L_i^{l_c} \\
&+ T_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} S_l f + (s-r-1) T_{(r+1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} f - \left( \sum_{b=1}^s \delta_{l_b}^{k_b} T_{(r+1)}^{k_1 \cdots \hat{k}_b \cdots k_s}_{l_1 \cdots l_b \cdots l_s} f \right) \\
&+ \left( \sum_{b \neq c} T_{(r+1)}^{k_1 \cdots \hat{k}_c \cdots \hat{k}_b \cdots k_s}_{l_1 \cdots l_c \cdots l_b \cdots l_s} \delta_{l_b}^{k_b} \right) f + \sum_{ij} T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} \bar{R}_{(n+1)\hat{ij}(n+1)} f,
\end{aligned} \tag{4-217}$$

$$\begin{aligned}
\frac{d}{dt} \widehat{T}_r^{k_1 \cdots k_s}_{l_1 \cdots l_s} &= [\widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f]_{,ij} - \sum_{ij} [\widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f]_{,j} \\
&- \sum_{ij} [\widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f]_{,i} + \sum_{ij} [\widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f]_{,ij} \\
&- \sum_i \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f \right]_{,ii} + \sum_i \left[ \frac{2(n+1-r)}{n} \widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f \right]_{,i} \\
&- \sum_i \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f \right]_{,i} + \sum_p \widehat{T}_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s, p} V^p \\
&- \sum_{c=1}^s \sum_i \widehat{T}_{(r)}^{k_1 \cdots \hat{k}_c \cdots k_s}_{l_1 \cdots l_c \cdots l_s} L_i^{k_c} - \sum_{c=1}^s \sum_j \widehat{T}_{(r)}^{k_1 \cdots \hat{k}_c \cdots k_s}_{l_1 \cdots l_c \cdots l_s} L_j^{l_c} \\
&- [(r+1) \widehat{T_{(r+1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} f] - \sum_{c=1}^s \sum_j [\widehat{T}_{(r)}^{k_1 \cdots \hat{k}_c \cdots k_s}_{l_1 \cdots l_c \cdots l_s} \hat{h}_{jl}^{k_c} f] \\
&+ r \widehat{T}_{(r)}^{k_1 \cdots k_s}_{l_1 \cdots l_s} H f + [\widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} H \hat{h}_{ij}^\gamma f] - \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} \hat{\sigma} f \right] \\
&+ [\widehat{T_{(r-1)}^{k_1 \cdots k_s}_{l_1 \cdots l_s}} \bar{R}_{(n+1)\hat{ij}(n+1)} f]
\end{aligned}$$

$$- \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} \bar{R}_{(n+1)(n+1)} f} \right]. \quad (4-218)$$

**推论 4.14:** 设  $x: M^n \rightarrow R^{n+1}(c)$  是超曲面, 设  $V = V^i e_i + fN$  是变分向量场, 那么我们有

$$\begin{aligned} \frac{d}{dt} T_{(r) l_1 \cdots l_s}^{k_1 \cdots k_s} &= \sum_{ij} [T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} f]_{,ij} + \sum_p T_{(r) l_1 \cdots l_s, p}^{k_1 \cdots k_s} V^p \\ &- \sum_{c=1}^s \sum_i T_{(r) l_1 \cdots l_c \cdots l_s}^{k_1 \cdots \hat{k}_c \cdots k_s} L_i^{k_c} + \sum_{c=1}^s \sum_i T_{(r) l_1 \cdots l_c \cdots l_s}^{k_1 \cdots \hat{k}_c \cdots k_s} L_i^{k_c} \\ &+ T_{(r) l_1 \cdots l_s}^{k_1 \cdots k_s} S_i f + (s-r-1) T_{(r+1) l_1 \cdots l_s}^{k_1 \cdots k_s} f \\ &- \left( \sum_{b=1}^s \delta_b^{k_b} T_{(r+1) l_1 \cdots l_b \cdots l_s}^{k_1 \cdots \hat{k}_b \cdots k_s} \right) f + \left( \sum_{b \neq c} T_{(r+1) l_1 \cdots l_c \cdots l_b \cdots l_s}^{k_1 \cdots \hat{k}_c \cdots k_s} \delta_b^{k_c} \right) f \\ &+ c(n+1-r-s) T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} f, \end{aligned} \quad (4-219)$$

$$\begin{aligned} \frac{d}{dt} \widehat{T_{r l_1 \cdots l_s}^{k_1 \cdots k_s}} &= \sum_{ij} [\widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} l_j f}]_{,ij} - \sum_{ij} [\widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} i f}]_{,j} \\ &- \sum_{ij} [\widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s}}]_{,i} + \sum_{ij} [\widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} j f}] \\ &- \sum_i \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} f} \right]_{,ii} + \sum_i \left[ \frac{2(n+1-r)}{n} \widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} f} \right]_{,i} \\ &- \sum_i \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} i f} \right] + \sum_p \widehat{T_{(r) l_1 \cdots l_s, p}^{k_1 \cdots k_s} V^p} \\ &- \sum_{c=1}^s \sum_i \widehat{T_{(r) l_1 \cdots l_c \cdots l_s}^{k_1 \cdots \hat{k}_c \cdots k_s} L_i^{k_c}} - \sum_{c=1}^s \sum_j \widehat{T_{(r) l_1 \cdots l_c \cdots l_s}^{k_1 \cdots \hat{k}_c \cdots k_s} L_j^{k_c}} \\ &- [(r+1) \widehat{T_{(r+1) l_1 \cdots l_s}^{k_1 \cdots k_s} f}] - \sum_{c=1}^s \sum_j [\widehat{T_{(r) l_1 \cdots l_c \cdots l_s}^{k_1 \cdots \hat{k}_c \cdots k_s} h_{ij} f}] \\ &+ r \widehat{T_{(r) l_1 \cdots l_s}^{k_1 \cdots k_s} H f} + [\widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} H h_{ij} f}] \\ &- \left[ \frac{(n+1-r)}{n} \widehat{T_{(r-1) l_1 \cdots l_s}^{k_1 \cdots k_s} \hat{\sigma} f} \right]. \end{aligned} \quad (4-220)$$

**推论 4.15:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 设  $V = V^i e_i + fN$  是变分向量场,  $r$  任意, 那么我们有

$$\begin{aligned} \frac{d}{dt} S_r &= \sum_{ij} (T_{(r) ij} f)_{,ij} - \sum_{ij} 2 [T_{(r) ij, j} f]_{,i} \\ &+ \sum_{ij} (T_{(r) ij, i} f) + \sum_p S_{r,p} V^p + S_r S_i f \\ &- (r+1) S_{r+1} f + \sum_{ij} T_{(r-1) ij} \bar{R}_{(n+1)ij(n+1)} f, \end{aligned} \quad (4-221)$$

$$\begin{aligned} \frac{d}{dt} \hat{S}_r &= (\widehat{T_{(r-1) ij} f})_{,ij} - 2 [\widehat{T_{(r-1) ij, j} f}]_{,i} + (\widehat{T_{(r-1) ij, i} f}) \\ &- \frac{(n+1-r)}{n} [\hat{S}_{r-1} f]_{,ii} + \frac{2(n+1-r)}{n} [\hat{S}_{r-1, i} f]_{,i} - \frac{(n+1-r)}{n} [\hat{S}_{r-1, ij} f] \end{aligned}$$

$$\begin{aligned}
& + \hat{S}_{r,p} V^p - (r+1) \hat{S}_{r+1} f + r \hat{S}_r Hf + \widehat{T_{(r-1)ij}} \hat{H} h_{ij} f - \frac{(n+1-r)}{n} \hat{S}_{r-1} \hat{\sigma} f \\
& + \widehat{T_{(r-1)ij}} \bar{R}_{(n+1)ij(n+1)} f - \frac{(n+1-r)}{n} \hat{S}_{r-1} \bar{R}_{(n+1)(n+1)} f. \quad (4-222)
\end{aligned}$$

**推论 4.16:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 设  $V = V^i e_i + fN$  是变分向量场,  $r$  任意, 那么我们有

$$\begin{aligned}
\frac{d}{dt} \int_M S_r dv &= \int_M \sum_{ij} (T_{(r)ij} f)_{,ij} + (r+1) S_{r+1} f \\
&+ \sum_{ij} T_{(r-1)ij} \bar{R}_{(n+1)ij(n+1)} f dv, \quad (4-223)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \int_M \hat{S}_r dv &= \int_M (\widehat{T_{(r-1)ij}} f)_{,ij} - \frac{(n+1-r)}{n} [\hat{S}_{r-1, i} f] \\
&- (r+1) \hat{S}_{r+1} f - (n-r) \hat{S}_r Hf + \widehat{T_{(r-1)ij}} \hat{H} h_{ij} f - \frac{(n+1-r)}{n} \hat{S}_{r-1} \hat{\sigma} f \\
&+ \widehat{T_{(r-1)ij}} \bar{R}_{(n+1)ij(n+1)} f - \frac{(n+1-r)}{n} \hat{S}_{r-1} \bar{R}_{(n+1)(n+1)} f dv. \quad (4-224)
\end{aligned}$$

**推论 4.17:** 设  $x: M^n \rightarrow R^{n+1}(c)$  是超曲面, 设  $V = V^i e_i + fN$  是变分向量场,  $r$  任意, 那么我们有

$$\begin{aligned}
\frac{d}{dt} S_r &= \sum_{ij} (T_{(r)ij} f)_{,ij} + \sum_p S_{r,p} V^p + S_r S_1 f \\
&- (r+1) S_{r+1} f + c(n-r+1) S_{r-1} f, \quad (4-225)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \hat{S}_r &= [\widehat{T_{(r-1)ij}} f]_{,ij} - 2 [\widehat{T_{(r-1)ij}} f]_{,j} + [\widehat{T_{(r-1)ij}} f]_{,i} \\
&- \frac{(n+1-r)}{n} [\hat{S}_{r-1} f]_{,i} + \frac{2(n+1-r)}{n} [\hat{S}_{r-1, i} f]_{,i} - \frac{(n+1-r)}{n} [\hat{S}_{r-1, i} f]_{,i} \\
&+ \hat{S}_{r,p} V^p - (r+1) \hat{S}_{r+1} f + r \hat{S}_r Hf + \widehat{T_{(r-1)ij}} \hat{H} h_{ij} f \\
&- \frac{(n+1-r)}{n} \hat{S}_{r-1} \hat{\sigma} f. \quad (4-226)
\end{aligned}$$

**推论 4.18:** 设  $x: M^n \rightarrow R^{n+1}(c)$  是紧致无边子流形, 设  $V = V^i e_i + fN$  是变分向量场,  $r$  是任意数, 那么我们有

$$\frac{d}{dt} \int_M S_r dv_i = \int_M - (r+1) S_{r+1} f + c(n-r+1) S_{r-1} f dv_i, \quad (4-227)$$

$$\begin{aligned}
\frac{d}{dt} \int_M \hat{S}_r dv &= \int_M (\widehat{T_{(r-1)ij}} f)_{,ij} - \frac{(n+1-r)}{n} [\hat{S}_{r-1, i} f]_{,i} - (r+1) \hat{S}_{r+1} f \\
&- (n-r) \hat{S}_r Hf + \widehat{T_{(r-1)ij}} \hat{H} h_{ij} f - \frac{(n+1-r)}{n} \hat{S}_{r-1} \hat{\sigma} f. \quad (4-228)
\end{aligned}$$

## 第5章 自伴算子的组合构造

从第二基本型出发,按照一定的规则,可以构造新的张量,这些张量在刻画某些特殊子流形和简化某些泛函的计算之中有巨大的应用,除此之外,这些特殊张量可以构造一些二阶微分算子,特别是一些自伴的二阶微分算子在子流形刚性定理和间隙定理的研究之中有重大价值.本章主要研究一种特殊的张量构造的特殊算子.参见文献[5,42].

### 5.1 自伴算子的定义

设  $\phi = \sum_{ij} \phi_{ij} \theta^i \otimes \theta^j$  是流形  $(M, ds^2)$  上的对称张量.定义 Cheng-Yau 微分算子  $\square$ :

$$\square f = \sum_{ij} \phi_{ij} f_{,ij}.$$

对于这个算子,容易得如下的定理.

**定理 5.1** (参见[28]): 设  $(M, ds^2)$  是紧致无边的,算子  $\square$  是自伴随的(在  $L^2$  中),当且仅当对任意  $i$

$$\sum_j \phi_{ij,j} = 0.$$

**证明:** 文献[28]假设函数  $f, g$  是光滑的,利用 Stokes 定理和分部积分公式,直接计算,我们有

$$\begin{aligned} \int_M \square f g \, dv &= \int_M \phi_{ij} f_{,ij} g \, dv = \int_M (\phi_{ij} f_{,i})_{,j} g - \phi_{ij,j} f_{,i} g \, dv \\ &= \int_M -\phi_{ij,i} f_{,j} g - \phi_{ij,j} f_{,i} g \, dv \\ &= \int_M -(\phi_{ij} f)_{,i} g_{,j} + \phi_{ij,i} f g_{,j} - \phi_{ij,j} f_{,i} g \, dv \\ &= \int_M (\phi_{ij} f g_{,ji} + \phi_{ij,i} f g_{,j} - \phi_{ij,j} f_{,i} g \, dv \\ &= \int_M \square g f + \phi_{ji,i} f g_{,j} - \phi_{ij,j} f_{,i} g \, dv. \end{aligned}$$

因此,根据函数和张量  $f, f_{,i}, g, g_{,j}$  的任意性,算子  $\square$  为自伴算子,当且仅当

$$\sum_j \varphi_{ij,j} = 0, \forall i.$$

我们列出一些自伴随算子的例子。

**例 5.1:** 最著名的例子自然是  $\Delta$  算子, 即  $\varphi_{ij} = \delta_{ij}$ .

**例 5.2:** 由第二 Bianchi 恒等式, 有  $\sum_j R_{ij,j} = \frac{1}{2} R_{,i}$ , 因此可以定义  $\varphi_{ij} = \frac{1}{2} R \delta_{ij} - R_{ij}$ . 实际上, 我们可以给出一个简洁的证明。

**证明** 我们知道 Bianchi 等式和对称等式

$$R_{ijkl, h} + R_{ijlh, k} + R_{ijhk, l} = 0, R_{ijkl} = R_{klij}.$$

所以根据定义和上面的等式, 我们有

$$\begin{aligned} I &= \sum_j R_{ij,j} = \sum_{jk} R_{ikhj,j} = \sum_{jk} R_{kjik,j} = \sum_{jk} -(R_{kjhj,i} + R_{kji,i,k}) \\ &= -R_{,i} - \sum_k R_{ik,k} = R_{,i} - I, I = \sum_j R_{ij,j} = \frac{1}{2} R_{,i}. \end{aligned}$$

**例 5.3:** 设对称张量  $a = \sum_{ij} a_{ij} \theta^i \otimes \theta^j$  满足 Codazzi 方程  $a_{ij,k} = a_{ik,j}$ , 则算子  $\varphi_{ij} = (\sum_k a_{kk}) \delta_{ij} - a_{ij}$  是自伴算子。我们有如下的推导:

$$\sum_j \varphi_{ij,j} = \sum_j (\sum_k a_{kk}) \delta_{ij} - \sum_j a_{ij,j} \quad (5-1)$$

$$\begin{aligned} &= (tr(a))_{,i} - \sum_j a_{ji,j} = (tr(a))_{,i} - \sum_j a_{ji,i} \\ &= (tr(a))_{,i} - (tr(a))_{,i} = 0. \end{aligned} \quad (5-2)$$

**例 5.4:** 设  $x: M \rightarrow R^{n+1}(c)$  是子流形,  $h_{ij}$  显然满足 Codazzi 方程, 算子  $\varphi_{ij} = nH\delta_{ij} - h_{ij}$  是自伴算子, 定义如下:

$$\square: C^\infty(M) \rightarrow C^\infty(M), \quad (5-3)$$

$$f \mapsto (nH\delta_{ij} - h_{ij})f_{ij} = nH\Delta f - \sum_{ij} h_{ij} f_{ij}. \quad (5-4)$$

**例 5.5:** 设  $x: M \rightarrow R^{n+p}(c)$  是子流形,  $h_{ij}^\alpha$  显然满足 Codazzi 方程, 算子  $\varphi_{ij}^\alpha = nH^\alpha \delta_{ij} - h_{ij}^\alpha$  对于固定的  $\alpha$  是自伴随的。定义如下:

$$\square^\alpha: C^\infty(M) \rightarrow C^\infty(TM), \quad (5-5)$$

$$f \mapsto (nH^\alpha \delta_{ij} - h_{ij}^\alpha) f_{ij} = nH^\alpha \Delta f - \sum_{ij} h_{ij}^\alpha f_{ij}, \quad (5-6)$$

$$\square: C^\infty(M) \rightarrow C^\infty(T^\perp M), \quad (5-7)$$

$$f \mapsto (nH^\alpha \delta_{ij} - h_{ij}^\alpha) f_{ij} e_\alpha = n\Delta f H - \sum_{ij} f_{ij} B_{ij}, \quad (5-8)$$

$$\square^\alpha: C^\infty(T^\perp M) \rightarrow C^\infty(TM), \quad (5-9)$$

$$\xi^\alpha e_\alpha \mapsto \Delta(nH^\alpha \xi^\alpha) - (h_{ij}^\alpha \xi^\alpha)_{,ij} = nH^\alpha \Delta \xi^\alpha - h_{ij}^\alpha \xi^\alpha_{,ij}. \quad (5-10)$$

**例 5.6:** 设  $x: M \rightarrow R^{n+1}(c)$  是子流形, Newton 变换  $T_{(r)j}^i$  显然散度为零, 算子

$\varphi_{ij} = T_{(r)j}^i$  自伴随的。

我们定义如下：

- $p=1, r$  任意

$$L_r: C^\infty(M) \rightarrow C^\infty(M), \quad (5-11)$$

$$f \rightarrow T_{(r)j}^i f_{ij}. \quad (5-12)$$

- $p=1, r$  任意

$$L_r^* = L_r, \int_M L_r^*(f) = \int_M L_r(f) = 0, \int_M f L_r(g) = - \int_M \langle T_{(r)} Df, Dg \rangle.$$

- $p=1, r$  任意

$$Q_r: C^\infty(M) \rightarrow C^\infty(M), \quad (5-13)$$

$$f \rightarrow T_{(r)j}^i f_{ij} + c(n-r) S_r f, \quad (5-14)$$

$$\text{i. e. } Q_r = L_r + c(n-r) S_{r \circ} \text{id}. \quad (5-15)$$

**例 5.7:** 设  $x: M \rightarrow R^{n+p}(c)$  是子流形,  $r$  是偶数, Newton 变换  $T_{(r)j}^i$  显然散度为零, 算子  $\varphi_{ij} = T_{(r)j}^i$  是自伴随的。我们定义如下：

- $p \geq 2, r$  为偶数

$$L_r: C^\infty(M) \rightarrow C^\infty(M), \quad (5-16)$$

$$f \rightarrow T_{(r)j}^i f_{ij}. \quad (5-17)$$

- $p \geq 2, r$  为偶数

$$L_r^* = L_r, \int_M L_r^*(f) = \int_M L_r(f) = 0, \int_M f L_r(g) = - \int_M \langle T_{(r)} Df, Dg \rangle.$$

- $p \geq 2, r$  为偶数

$$Q_r: C^\infty(M) \rightarrow C^\infty(M), \quad (5-18)$$

$$f \rightarrow T_{(r)j}^i f_{ij} + c(n-r) S_r f, \quad (5-19)$$

$$\text{i. e. } Q_r = L_r + c(n-r) S_{r \circ} \text{id}. \quad (5-20)$$

**例 5.8:** 设  $x: M \rightarrow R^{n+p}(c)$  是子流形, 设  $r$  是奇数, Newton 变换  $T_{(r)j}^\alpha$  显然散度为零, 算子  $\varphi_{ij}^\alpha = T_{(r)j}^\alpha$  对于固定的  $\alpha$  是自伴随的。定义如下：

- $p \geq 2, r$  为奇数

$$L_r^\alpha: C^\infty(M) \rightarrow C^\infty(M), \quad (5-21)$$

$$f \rightarrow T_{(r)j}^\alpha f_{ij}, \quad (5-22)$$

$$L_r: C^\infty(M) \rightarrow C^\infty(T^\perp M), \quad (5-23)$$

$$f \rightarrow T_{(r)j}^\alpha f_{ij} e_\alpha, \quad (5-24)$$

$$\int_M L_r^\alpha f = \int_M \langle L_r f, e_\alpha \rangle = 0. \quad (5-25)$$

- $p \geq 2, r$  为奇数

$$L_r^*: C^\infty(T^\perp M) \rightarrow C^\infty(M), \quad (5-26)$$



$$\xi^\alpha e_\alpha \rightarrow T_{(r)}^\alpha \xi_{,ij}^\alpha, \quad (5-27)$$

$$\int_M L_r^*(\xi^\alpha e_\alpha) = 0. \quad (5-28)$$

•  $p \geq 2$ ,  $r$  为奇数

$$Q_r^\alpha: C^\infty(M) \rightarrow C^\infty(M), \quad (5-29)$$

$$f \rightarrow T_{(r)}^\alpha f_{,ij} + c(n-r) \langle \vec{S}_r, e_\alpha \rangle f, \quad (5-30)$$

$$\text{i. e. } Q_r^\alpha = L_r^\alpha + c(n-r) \langle \vec{S}_r, e_\alpha \rangle \text{id}, \quad (5-31)$$

$$Q_r: C^\infty(M) \rightarrow C^\infty(T^\perp M), \quad (5-32)$$

$$f \rightarrow T_{(r)}^\alpha f_{,ij} e_\alpha + c(n-r) f \cdot \vec{S}_r, \quad (5-33)$$

$$\text{i. e. } Q_r = L_r + c(n-r) \vec{S}_r \circ \text{id}. \quad (5-34)$$

## 5.2 曲率模长和 Willmore 不变量的计算

我们都知道, 子流形第二基本型长度函数和 Willmore 不变量分别定义为  $S = \sum_{ij\alpha} (h_{ij}^\alpha)^2$  和  $\rho = S - nH^2$ , 它们的二阶协变导数的计算是非常有用的。我们分别计算之。

• 在一般流形之中且  $p \geq 2$  时

$$S_{,kl} = \sum_{ij\alpha} 2(h_{ij}^\alpha h_{ij,k}^\alpha)_{,l} = \sum_{ij\alpha} 2h_{ij}^\alpha h_{ij,kl}^\alpha + \sum_{ij\alpha} 2h_{ij,k}^\alpha h_{ij,l}^\alpha \quad (5-35)$$

$$= \sum_{ij\alpha} 2h_{ij}^\alpha ((h_{ij,k}^\alpha - h_{ik,j}^\alpha)_{,l} + (h_{ik,jl}^\alpha - h_{ik,lj}^\alpha) + (h_{kl,i}^\alpha - h_{kl,i}^\alpha)_{,j} + h_{kl,ij}^\alpha) + 2 \sum_{ij\alpha} h_{ij,k}^\alpha h_{ij,l}^\alpha \quad (5-36)$$

$$= \sum_{ij\alpha} -2h_{ij}^\alpha \bar{R}_{ijk,l}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha \bar{R}_{kli,j}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha h_{kl,ij}^\alpha + \sum_{ij\alpha} 2h_{ij,k}^\alpha h_{ij,l}^\alpha + 2\left\{ \sum_{ij\alpha} h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjl}^\alpha + \sum_{ij\alpha} h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{knpj}^\alpha + \sum_{ij\alpha\beta} h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jl}^\alpha + \sum_{ij\alpha\beta} (h_{ij}^\alpha h_{il}^\beta h_{kp}^\beta h_{pj}^\alpha - h_{ij}^\alpha h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta) + \sum_{ij\alpha\beta} (h_{ij}^\alpha h_{ip}^\alpha h_{pj}^\beta h_{kl}^\beta - h_{ij}^\alpha h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) + \sum_{ij\alpha\beta} (h_{ij}^\alpha h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta - h_{ij}^\alpha h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta) \right\}, \quad (5-37)$$

$$\rho_{,kl} = S_{,kl} - \sum_\alpha 2nH_{,k}^\alpha H_{,l}^\alpha - \sum_\alpha 2nH^\alpha H_{,kl}^\alpha \quad (5-38)$$

$$= \sum_{ij\alpha} -2h_{ij}^\alpha \bar{R}_{ijk,l}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha \bar{R}_{kli,j}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha h_{kl,ij}^\alpha + \sum_{ij\alpha} 2h_{ij,k}^\alpha h_{ij,l}^\alpha + 2\left\{ \sum_{ij\alpha} h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjl}^\alpha + \sum_{ij\alpha} h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{knpj}^\alpha + \sum_{ij\alpha\beta} h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jl}^\alpha + \sum_{ij\alpha\beta} (h_{ij}^\alpha h_{il}^\beta h_{kp}^\beta h_{pj}^\alpha - h_{ij}^\alpha h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta) + \sum_{ij\alpha\beta} (h_{ij}^\alpha h_{ip}^\alpha h_{pj}^\beta h_{kl}^\beta - h_{ij}^\alpha h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) \right\}$$

$$+ \sum_{ijp\alpha\beta} (h_{ij}^{\alpha} h_{ik}^{\beta} h_{jp}^{\beta} h_{pl}^{\alpha} - h_{ij}^{\alpha} h_{ik}^{\beta} h_{jp}^{\alpha} h_{pl}^{\beta}) \} \quad (5-39)$$

$$- \sum_{\alpha} 2n H_{,k}^{\alpha} H_{,l}^{\alpha} - \sum_{\alpha} 2n H^{\alpha} H_{,kl}^{\alpha}, \quad (5-40)$$

$$\begin{aligned} \Delta S = \sum_k S_{,kk} &= \sum_{ijk\alpha} -2h_{ij}^{\alpha} \bar{R}_{ijk}^{\alpha} + \sum_{ijk\alpha} 2h_{ij}^{\alpha} \bar{R}_{kkl, j}^{\alpha} + \sum_{ij\alpha} 2nh_{ij}^{\alpha} H_{,ij}^{\alpha} + 2|Dh|^2 \\ &+ 2\{ \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpyk} + \sum_{ijk\alpha\beta} h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk} \} \\ &- \sum_{\alpha \neq \beta} 2N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) \\ &+ \sum_{\alpha\beta} 2n S_{\alpha\alpha\beta\beta} H^{\beta} - 2(S_{\alpha\beta})^2, \end{aligned} \quad (5-41)$$

$$\begin{aligned} \Delta \rho = \sum_{ijk\alpha} -2h_{ij}^{\alpha} \bar{R}_{ijk}^{\alpha} + \sum_{ijk\alpha} 2h_{ij}^{\alpha} \bar{R}_{kkl, j}^{\alpha} + \sum_{ij\alpha} 2nh_{ij}^{\alpha} H_{,ij}^{\alpha} + 2|Dh|^2 \\ + 2\{ \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpyk} + \sum_{ijk\alpha\beta} h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk} \} \\ - \sum_{\alpha \neq \beta} 2N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) + \sum_{\alpha\beta} 2n S_{\alpha\alpha\beta\beta} H^{\beta} - 2(S_{\alpha\beta})^2 \end{aligned} \quad (5-42)$$

$$- 2n |\nabla H|^2 - \sum_{\alpha} 2n H^{\alpha} \Delta H^{\alpha}, \quad (5-43)$$

• 在一般流形之中且  $p=1$  时

$$\begin{aligned} S_{,kl} &= \sum_{ij} 2h_{ij} \bar{R}_{(n+1)ijk, l} - \sum_{ij} 2h_{ij} \bar{R}_{(n+1)kkl, j} + \sum_{ij} 2h_{ij} h_{kl, ij} + \sum_{ij} 2h_{ij, k} h_{ij, l} \\ &+ 2\{ \sum_{ijp} h_{ij} h_{pk} \bar{R}_{ipjl} + \sum_{ijp} h_{ij} h_{ip} \bar{R}_{kpjl} - S \sum_p h_{kp} h_{pl} \\ &+ \sum_{ijp} (h_{ij} h_{il} h_{kp} h_{pj} + h_{ij} h_{ip} h_{pj} h_{kl} - h_{ij} h_{ip} h_{pl} h_{jk}) \}, \end{aligned} \quad (5-44)$$

$$\begin{aligned} \rho_{,kl} &= S_{,kl} - 2n H_{,k} H_{,l} - 2n H H_{,kl} \\ &= \sum_{ij} 2h_{ij} \bar{R}_{(n+1)ijk, l} - \sum_{ij} 2h_{ij} \bar{R}_{(n+1)kkl, j} + \sum_{ij} 2h_{ij} h_{kl, ij} + \sum_{ij} 2h_{ij, k} h_{ij, l} \\ &+ 2\{ \sum_{ijp} h_{ij} h_{pk} \bar{R}_{ipjl} + \sum_{ijp} h_{ij} h_{ip} \bar{R}_{kpjl} - S \sum_p h_{kp} h_{pl} \\ &+ \sum_{ijp} (h_{ij} h_{il} h_{kp} h_{pj} + h_{ij} h_{ip} h_{pj} h_{kl} - h_{ij} h_{ip} h_{pl} h_{jk}) \} - 2n H_{,k} H_{,l} - 2n H H_{,kl}, \end{aligned} \quad (5-45)$$

$$\begin{aligned} \Delta S = \sum_k S_{,kk} &= \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)ijk, k} - \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)kkl, j} + \sum_{ij} 2nh_{ij} H_{,ij} + 2|Dh|^2 \\ &+ \sum_{ijkl} 2h_{ij} h_{kl} \bar{R}_{iljk} + \sum_{ijkl} 2h_{ij} h_{il} \bar{R}_{jkkd} - 2S^2 + 2nP_3 H, \end{aligned} \quad (5-47)$$

$$\Delta \rho = \Delta S - 2n |\nabla H|^2 - 2n H \Delta H \quad (5-48)$$

$$\begin{aligned} &= \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)ijk, k} - \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)kkl, j} + \sum_{ij} 2nh_{ij} H_{,ij} + 2|Dh|^2 + \sum_{ijkl} 2h_{ij} h_{kl} \bar{R}_{iljk} \\ &+ \sum_{ijkl} 2h_{ij} h_{il} \bar{R}_{jkkd} - 2S^2 + 2nP_3 H - 2n |\nabla H|^2 - 2n H \Delta H. \end{aligned} \quad (5-49)$$

- 在空间形式之中且  $p \geq 2$  时

$$\begin{aligned}
 S_{,kl} = & \sum_{ij\alpha} 2h_{ij}^{\alpha} h_{kl,ij}^{\alpha} + \sum_{ij\alpha} 2h_{ij,k}^{\alpha} h_{ij,l}^{\alpha} + 2 \left\{ \sum_{\alpha} -cnH^{\alpha} h_{kl}^{\alpha} + c\delta_{kl}S \right. \\
 & + \sum_{ij\rho\alpha\beta} (h_{ij}^{\alpha} h_{il}^{\beta} h_{kp}^{\alpha} h_{pj}^{\beta} - h_{ij}^{\alpha} h_{ij}^{\beta} h_{kp}^{\alpha} h_{pl}^{\beta}) + \sum_{ij\rho\alpha\beta} (h_{ij}^{\alpha} h_{ip}^{\alpha} h_{pj}^{\beta} h_{kl}^{\beta} - h_{ij}^{\alpha} h_{ip}^{\alpha} h_{pl}^{\beta} h_{jk}^{\beta}) \\
 & \left. + \sum_{ij\rho\alpha\beta} (h_{ij}^{\alpha} h_{ik}^{\beta} h_{jp}^{\beta} h_{pl}^{\alpha} - h_{ij}^{\alpha} h_{ik}^{\beta} h_{jp}^{\alpha} h_{pl}^{\beta}) \right\}, \quad (5-50)
 \end{aligned}$$

$$\rho_{,kl} = S_{,kl} - \sum_{\alpha} 2nH^{\alpha}_{,k} H^{\alpha}_{,l} - \sum_{\alpha} 2nH^{\alpha} H^{\alpha}_{,kl} \quad (5-51)$$

$$\begin{aligned}
 = & \sum_{ij\alpha} 2h_{ij}^{\alpha} h_{kl,ij}^{\alpha} + \sum_{ij\alpha} 2h_{ij,k}^{\alpha} h_{ij,l}^{\alpha} + 2 \left\{ \sum_{\alpha} -cnH^{\alpha} h_{kl}^{\alpha} + c\delta_{kl}S \right. \\
 & + \sum_{ij\rho\alpha\beta} (h_{ij}^{\alpha} h_{il}^{\beta} h_{kp}^{\alpha} h_{pj}^{\beta} - h_{ij}^{\alpha} h_{ij}^{\beta} h_{kp}^{\alpha} h_{pl}^{\beta}) + \sum_{ij\rho\alpha\beta} (h_{ij}^{\alpha} h_{ip}^{\alpha} h_{pj}^{\beta} h_{kl}^{\beta} - h_{ij}^{\alpha} h_{ip}^{\alpha} h_{pl}^{\beta} h_{jk}^{\beta}) \\
 & \left. + \sum_{ij\rho\alpha\beta} (h_{ij}^{\alpha} h_{ik}^{\beta} h_{jp}^{\beta} h_{pl}^{\alpha} - h_{ij}^{\alpha} h_{ik}^{\beta} h_{jp}^{\alpha} h_{pl}^{\beta}) \right\} - \sum_{\alpha} 2nH^{\alpha}_{,k} H^{\alpha}_{,l} - \sum_{\alpha} 2nH^{\alpha} H^{\alpha}_{,kl}, \quad (5-52)
 \end{aligned}$$

$$\begin{aligned}
 \Delta S = \sum_k S_{,kk} = & \sum_{ij\alpha} 2nh_{ij}^{\alpha} H^{\alpha}_{,ij} + 2 | Dh |^2 + 2ncS - 2n^2 cH^2 \\
 & - \sum_{\alpha \neq \beta} 2N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta} H^{\beta} - 2(S_{\alpha\beta})^2, \quad (5-53)
 \end{aligned}$$

$$\Delta \rho = \Delta S - 2n | \vec{\nabla H} |^2 - \sum_{\alpha} 2nH^{\alpha} \Delta H^{\alpha} \quad (5-54)$$

$$= \sum_{ij\alpha} 2nh_{ij}^{\alpha} H^{\alpha}_{,ij} + 2 | Dh |^2 + 2ncS - 2n^2 cH^2 \quad (5-55)$$

$$- \sum_{\alpha \neq \beta} 2N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta} H^{\beta} - 2(S_{\alpha\beta})^2 \quad (5-56)$$

$$- 2n | \vec{\nabla H} |^2 - \sum_{\alpha} 2nH^{\alpha} \Delta H^{\alpha}. \quad (5-57)$$

- 在空间形式之中且  $p = 1$  时

$$\begin{aligned}
 S_{,kl} = & \sum_{ij} 2h_{ij} h_{kl,ij} + \sum_{ij} 2h_{ij,k} h_{ij,l} - 2cnHh_{kl} + 2c\delta_{kl}S - 2S \sum_p h_{kp} h_{pl} \\
 & + \sum_{ijp} 2(h_{ij} h_{il} h_{kp} h_{pj} + h_{ij} h_{ip} h_{pj} h_{kl} - h_{ij} h_{ip} h_{pl} h_{jk}), \quad (5-58)
 \end{aligned}$$

$$\rho_{,kl} = S_{,kl} - 2nH_{,k} H_{,l} - 2nHH_{,kl} \quad (5-59)$$

$$\begin{aligned}
 = & \sum_{ij} 2h_{ij} h_{kl,ij} + \sum_{ij} 2h_{ij,k} h_{ij,l} - 2cnHh_{kl} + 2c\delta_{kl}S - 2S \sum_p h_{kp} h_{pl} \\
 & + \sum_{ijp} 2(h_{ij} h_{il} h_{kp} h_{pj} + h_{ij} h_{ip} h_{pj} h_{kl} - h_{ij} h_{ip} h_{pl} h_{jk}) - 2nH_{,k} H_{,l} - 2nHH_{,kl}, \quad (5-60)
 \end{aligned}$$

$$\begin{aligned}
 \Delta S = \sum_k S_{,kk} = & \sum_{ij} 2nh_{ij} H_{,ij} + 2 | Dh |^2 \\
 & - 2n^2 cH^2 + 2ncS - 2S^2 + 2nHP_3, \quad (5-61)
 \end{aligned}$$

$$\Delta \rho = \Delta S - 2n | \nabla H |^2 - 2nH\Delta H$$

$$\begin{aligned}
&= \sum_{ij} 2nh_{ij}H_{,ij} + 2|Dh|^2 - 2n^2cH^2 \\
&\quad + 2ncS - 2S^2 + 2nHP_3 - 2n|\nabla H|^2 - 2nH\Delta H.
\end{aligned} \tag{5-62}$$

### 5.3 位置向量、切向量和法向量的计算

空间形式之中子流形位置向量、切向量和法向量的二阶协变微分的计算往往可以给出很多特殊子流形的微分刻画。

#### • 位置向量 $x$ 的协变导数

固定一个向量  $a$ , 定义函数  $f = \langle x, a \rangle$ , 根据函数协变导数的定义, 我们有

$$df = f_{,i}\theta^i = \langle dx, a \rangle = \langle e_i, a \rangle \theta^i, \tag{5-63}$$

$$\begin{aligned}
f_{,ij}\theta^j &= df_i - f_p\phi_i^p = d\langle e_i, a \rangle - \langle e_p, a \rangle \phi_i^p \\
&= \langle \phi_i^p e_p + \phi_i^\alpha e_\alpha - c\theta^i x, a \rangle - \langle e_p, a \rangle \phi_i^p \\
&= \langle h_{ij}^\alpha e_\alpha \theta^j - c\delta_{ij} x \theta^j, a \rangle,
\end{aligned} \tag{5-64}$$

$$x_{,i} = e_i, \quad x_{,ij} = h_{ij}^\alpha e_\alpha - c\delta_{ij} x, \quad \Delta x = n \vec{H} - ncx, \tag{5-65}$$

$$L_r x = (r+1) \vec{S}_{r+1} - c(n-r) S_r x. \tag{5-66}$$

#### • 切向量 $e_i$ 的协变导数

固定一个向量  $a$ , 定义向量场  $\eta = \langle e_i, a \rangle e_i = \eta^i e_i$ , 根据向量场的定义, 我们有

$$\begin{aligned}
\eta_{,j}^i \theta^j &= d\eta^i + \eta^p \phi_p^i = d\langle e_i, a \rangle + \langle e_p, a \rangle \phi_p^i \\
&= \langle \phi_i^p e_p + \phi_i^\alpha e_\alpha - c\theta^i x, a \rangle + \langle e_p, a \rangle \phi_p^i \\
&= (h_{ij}^\alpha \langle e_\alpha, a \rangle - c\delta_{ij} \langle x, a \rangle) \theta^j,
\end{aligned} \tag{5-67}$$

$$e_{i,j} = h_{ij}^\alpha e_\alpha - c\delta_{ij} x, \quad e_{i,jk} = h_{ij,k}^\alpha e_\alpha - h_{ij}^\alpha h_{kp}^\alpha e_p - c\delta_{ij} e_k, \tag{5-68}$$

$$\Delta e_i = \sum_\alpha n H_{,i}^\alpha e_\alpha - \sum_{jk\alpha} h_{ij}^\alpha h_{jk}^\alpha e_k - c e_i. \tag{5-69}$$

#### • 法向量 $e_\alpha$ 的协变导数

固定一个向量  $a$ , 定义向量场  $\xi = \langle e_\alpha, a \rangle e_\alpha = \xi^\alpha e_\alpha$ , 根据向量场协变导数的定义, 我们有

$$\begin{aligned}
\xi_{,i}^\alpha \theta^i &= d\xi^\alpha + \xi^\beta \phi_\beta^\alpha = d\langle e_\alpha, a \rangle + \langle e_\beta, a \rangle \phi_\beta^\alpha \\
&= \langle \phi_\alpha^p e_p + \phi_\alpha^\beta e_\beta, a \rangle - \langle e_\beta, a \rangle \phi_\alpha^\beta = -h_{ij}^\alpha e_j,
\end{aligned} \tag{5-70}$$

$$e_{\alpha,i} = -h_{ip}^\alpha e_p, \quad e_{\alpha,ij} = -h_{ip,j}^\alpha e_p - h_{ip}^\alpha h_{j\beta}^\alpha e_\beta + c h_{ij}^\alpha x, \tag{5-70}$$

$$\Delta e_\alpha = -n \sum_i H_{,i}^\alpha e_i - \sum_\beta \sigma_{\alpha\beta} e_\beta + nc H^\alpha x, \tag{5-71}$$

$$\begin{aligned}
L_r^* (\langle a, e_\alpha \rangle e_\alpha) &= -\langle DS_{r+1}, a \rangle - S_{r+1} \langle \vec{S}_1, a \rangle + (r+2) \langle \vec{S}_{r+2}, a \rangle \\
&\quad + c(r+1) S_{r+1} \langle x, a \rangle.
\end{aligned} \tag{5-72}$$

## 第 6 章 曲率模长泛函的定义

前面五章我们充分给出了预备知识。从本章开始, 我们开始进入本书的主题部分, 曲率模长泛函的变分法研究。为了所发展出来的定理的适用的广泛性, 我们定义了一类与曲率模长相关的泛函  $GD_{(n,F)}$ , 同时为了研究其在微小摄动下的演变规律, 与之对应的, 我们定义了  $GD_{(n,F,\epsilon)}$  泛函。特别的, 当  $F$  取不同的典型函数的时候, 我们定义了一些特殊的  $GD$  类泛函。

### 6.1 曲率模长泛函的定义

在子流形几何之中, 通过第二基本型

$$B = h_{ij}^\alpha e_\alpha \otimes \theta^i \otimes \theta^j.$$

我们可以定义第二基本型曲率模长为

$$S = \sum_{\alpha ij} (h_{ij}^\alpha)^2.$$

简称为曲率模长。显然, 曲率模长  $S$  满足如下性质

- 非负性

曲率模长  $S$  非负, 即是  $S(Q) \geq 0, \forall Q \in M$ ;

- 零点即测地点

曲率模长  $S$  的零点即为测地点, 即是  $S(Q) = 0$  当且仅当  $Q$  是  $M$  的测地点;

- 有界性

因为  $M$  是紧致无边流形, 所以曲率模长  $S$  可被一个与流形  $M$  有关的正常数  $C_M$  控制, 即是  $0 \leq S \leq C_M$ 。极小子流形的研究是历史悠久的课题。在代数上, 它由

$$H^\alpha = \frac{1}{n} \sum_i h_{ii}^\alpha \equiv 0, \forall \alpha$$

来刻画。在变分法上, 它由体积泛函来刻画

$$\text{Vol}(x) = \int_M dv.$$

Simons 在其著名的子流形的论文之中利用特殊的自伴算子作用于曲率模长  $S$ , 并结合极小子流形方程和精巧的矩阵不等式, 得到了极小子流形的曲率模长的间隙现象。

$$\int_M S(S - \frac{n}{2 - \frac{1}{p}}) dv \geq 0.$$

根据此积分不等式, 容易得到当

$$0 \leq S \leq \frac{n}{2 - \frac{1}{p}}$$

时. 曲率模长  $S$  只能取到端点的两个值, 即是

$$S=0 \text{ 或者 } S = \frac{n}{2 - \frac{1}{p}}.$$

Lawson 从上面的结论出发, 对超曲面定出了  $S = n$  的极小子流形, 为 Clifford 环面. Chern, do Carmo, Kobayashi 则对高余维数情形定出了  $S = \frac{n}{2 - \frac{1}{p}}$  的极小子流

形, 为 Clifford 环面和 Veronese 曲面. Chern, do Carmo, Kobayashi 讨论的过程分为四步. 第一步, 利用特殊的自伴算子作用于曲率模长  $S$ , 利用极小子流形的代数方程和张量协变导数交换公式化简表达式; 第二步, 利用精巧的矩阵不等式估计得到的表达式; 第三步, 利用子流形结构方程讨论几何量的刚性; 第四步, 利用 Frobenius 定理得到子流形分解定理.

李安民和李济民对第二步中的矩阵不等式作了重要改进, 得到了李安民类型的间隙定理.

受此启发, 我们可以考虑所谓的  $GD_{(n, F)}$  泛函, 其临界点称为  $GD_{(n, F)}$  子流形. 为了叙述精确, 我们需要设定两个集合

$$T_1 = \{x: M^n \rightarrow N^{n+p}, M \text{ 是无测地点子流形}\}$$

$$T_2 = \{x: M^n \rightarrow N^{n+p}, M \text{ 是一般子流形}\}$$

根据集合  $T_1, T_2$  的定义, 我们精确定义函数  $F$  满足

$$F: (0, \infty) \text{ 或者 } [0, \infty) \rightarrow \mathbb{R}, u \rightarrow F(u).$$

并且满足

$$F \in C^3(0, \infty) \text{ 或者 } C^3[0, \infty).$$

当  $M$  是无测地点子流形时, 我们对函数  $F$  的要求为

$$F \in C^3(0, \infty), F: (0, \infty) \rightarrow \mathbb{R}, u \rightarrow F(u).$$

当  $M$  为一般子流形时, 我们对函数  $F$  的要求为

$$F \in C^3[0, \infty), F: [0, \infty) \rightarrow \mathbb{R}, u \rightarrow F(u).$$

由集合  $T_1, T_2$  的定义和函数  $F$  的选择, 我们可以定义新的两个集合

$$T_{1,1} = \{(M, F): M \text{ 是无测地点子流形}, F \in C^3(0, \infty)\}$$

$$T_{2,2} = \{(M, F): M \text{ 是一般子流形}, F \in C^3[0, \infty)\}$$

我们对于集合  $T_1$  或者集合  $T_2$  中的子流形, 我们分别选取  $T_{1,1}$  和  $T_{2,2}$  中的函数  $F$  来定义  $GD_{(n,F)}$  泛函为

$$GD_{(n,F)}(x) = \int_M F(S) dv.$$

为了研究  $GD_{(n,F)}$  泛函的微小扰动之下的变化规律, 我们进一步考虑  $GD_{(n,F,\epsilon)}$

$$GD_{(n,F,\epsilon)} = \int_M F(S + \epsilon) dv.$$

特别的, 对于各种不同的函数, 我们可以定义很多具体的泛函, 这些泛函可以丰富子流形的研究。抽象函数  $F$  的形式多种多样, 我们不可能一一考虑清楚, 我们只需要考虑几种典型的函数, 包括: 幂函数, 指数函数, 对数函数, 三角函数。下面我们逐一介绍。

## 6.2 特殊的曲率模长泛函

幂函数是一种典型函数。因此, 当  $F(u) = u^r$  时, 我们可以定义幂函数曲率模长泛函为

$$GD_{(n,r)} = \int_M S^r dv.$$

其临界点称为  $GD_{(n,r)}$  子流形。对于是否具有测地点的子流形, 指数  $r$  的取值很不相同。如前文所述集合定义为

$$T_1 = \{x: M^n \rightarrow N^{n+p}, M \text{ 是无测地点子流形}\},$$

$$T_2 = \{x: M^n \rightarrow N^{n+p}, M \text{ 是一般子流形}\}.$$

显然的, 对于上面不同的集合, 指数  $r$  的取值为

- 当  $M \in T_1$  时, 指数  $r$  的取值为

$$r \in \mathbb{R}.$$

- 当  $M \in T_2$  时, 指数  $r$  的取值为

$$r = 1, 2, \text{ 或者, } \in [3, \infty).$$

我们记为集合

$$T_{1,1} = \{(M, r): M \text{ 无测地线子流形}, r \in \mathbb{R}\},$$

$$T_{2,2} = \{(M, r): M \text{ 是一般子流形}, r = 1, 2, \text{ 或者, } \in [3, \infty)\}.$$

这样取值的目的是为了使得计算泛函  $GD_{(n,r)}$  的第一变分公式有意义。自然当  $r > 0$  时, 对于  $T_2$  之中的子流形, 泛函  $GD_{(n,r)}$  在积分上是有意义的, 但是通过变分公式的计算, 我们知道对于某些取值不一定有意义。

通过对幂函数曲率模长泛函的微小扰动, 可以实现对比研究, 我们定义  $GD_{(n,r,\epsilon)}$  泛函为

$$GD_{(n, r, \epsilon)} = \int_M (S + \epsilon)^r dv, \quad \forall \epsilon > 0.$$

因为曲率模长  $S \geq 0$ , 所以对于任意的  $\epsilon > 0$ , 我们都有  $S + \epsilon > 0$ . 因此上面的积分泛函  $GD_{(n, r, \epsilon)}$  对任意的子流形  $M \in T_1 \cup T_2$  都是有意义的.

指数函数是一种重要的基本函数, 它具有如下典型性质

$$e^S = 1 + S + \frac{1}{2!}S^2 + \cdots + \frac{1}{m!}S^m + \cdots$$

因此, 指数函数是对幂函数的某种意义上的线性组合, 体现了平均效应. 因此我们研究如下的泛函

$$GD_{(n, E)} = \int_M e^S dv$$

泛函的临界点称为  $GD_{(n, E)}$  子流形. 对于指数函数, 因为  $e^{S+\epsilon} = e^\epsilon e^S$ , 所以定义指数函数曲率模长扰动泛函是没有意义的, 故在此不予定义.

对数函数是一种基本函数, 它具有如下性质

$$\log(S) = \log(1 + S - 1) = (S - 1) - \frac{1}{2}(S - 1)^2 + \cdots + \frac{(-1)^{n-1}}{n}(S - 1)^n + \cdots$$

因此, 对数函数是对幂函数某种意义上的交错组合. 我们研究如下泛函

$$GD_{(n, \log)} = \int_M \log(S) dv, \quad M \in T_1$$

泛函的临界点称为  $GD_{(n, \log)}$  子流形. 同样的, 我们可以定义扰动的对数函数型曲率模长泛函为

$$GD_{(n, \log)} = \int_M \log(S + \epsilon) dv, \quad M \in T_1 \cup T_2.$$

三角函数是多种多样的, 我们只关注一种三角函数

$$\sin(S) = S - \frac{1}{3!}S^3 + \cdots + \frac{(-1)^n}{(2n+1)!}S^{2n+1} + \cdots$$

是奇数次方幂函数的组合. 我们研究

$$GD_{(n, \sin)} = \int_M \sin(S) dv$$

泛函的临界点称为  $GD_{(n, \sin)}$  子流形. 同样的, 我们可以定义扰动的三角函数型曲率模长泛函为

$$GD_{(n, \sin, \epsilon)} = \int_M \sin(S + \epsilon) dv.$$

综上, 我们在本书之中研究如下五类曲率模长泛函.

- 抽象函数型  $GD_{(n, F)}$  和  $GD_{(n, F, \epsilon)}$  泛函

$$GD_{(n, F)} = \int_M F(S) dv, \quad GD_{(n, F, \epsilon)} = \int_M F(S + \epsilon) dv. \quad (6-1)$$

研究此泛函的目的在于对各种零散的曲率模长泛函进行统一处理, 得到较为抽象



的结论。

- 幂函数型  $GD_{(n, r)}$  和  $GD_{(n, r, \epsilon)}$  泛函

$$GD_{(n, r)} = \int_M S^r dv, \quad GD_{(n, r, \epsilon)} = \int_M (S + \epsilon)^r dv. \quad (6-2)$$

- 指数函数型  $GD_{(n, E)}$  泛函

$$GD_{(n, E)} = \int_M e^S dv. \quad (6-3)$$

- 对数函数型  $GD_{(n, \log)}$  和  $GD_{(n, \log, \epsilon)}$  泛函

$$GD_{(n, \log)} = \int_M \log(S) dv, \quad GD_{(n, \log, \epsilon)} = \int_M \log(S + \epsilon) dv. \quad (6-4)$$

- 三角函数型  $GD_{(n, \sin)}$  和  $GD_{(n, \sin, \epsilon)}$  泛函

$$GD_{(n, \sin)} = \int_M \sin(S) dv, \quad GD_{(n, \sin, \epsilon)} = \int_M \sin(S + \epsilon) dv. \quad (6-5)$$

构造后面四类 Willmore 类泛函的目的在于基于  $F$  的具体表达式, 对四种典型函数的泛函进行样本性研究。

## 第7章 泛函的第一变分

在第6章,我们构造了很多曲率模长泛函,本章我们利用第3章和第4章的基本变分公式来计算这些泛函的第一变分,这是对曲率模长泛函进行研究的基础。

### 7.1 $GD_{(n,F)}$ 泛函的第一变分公式

因为  $GD_{(n,F)}$  泛函的抽象性,为了使得计算结果适用于较大的范围,本节我们计算  $GD_{(n,F)}$  泛函和  $GD_{(n,F,\epsilon)}$  泛函的变分方程。为此,我们需要如下引理。

**引理 7.1:** 假设  $x:M^n \rightarrow N^{n+p}$  是子流形,  $V = \sum_i V^i e_i + \sum_\alpha V^\alpha e_\alpha$  是浸入映射的变分向量场,对于子流形  $M$  的体积微元,我们有

$$\frac{\partial dv}{\partial t} = \left( \sum_i V^i_{,i} - n \sum_\alpha H^\alpha V^\alpha \right) dv. \quad (7-1)$$

**证明:** 由定理 3.3 立即可得。

**引理 7.2:** 假设  $x:M^n \rightarrow N^{n+p}$  是子流形,  $V = \sum_i V^i e_i + \sum_\alpha V^\alpha e_\alpha$  是浸入映射的变分向量场,对于子流形  $M$  的 Willmore 不变量  $\rho$ , 我们有

$$\frac{\partial S}{\partial t} = \sum 2h_{ij}^\alpha V^\alpha_{,ij} + \sum_i S_{,i} V^i + \sum_{\alpha\beta} 2S_{\alpha\alpha\beta} V^\beta - \sum 2h_{ij}^\alpha \bar{R}_{ij\alpha}^\beta V^\beta. \quad (7-2)$$

**证明:** 由推论 3.4 立刻可得。

利用上面的两个引理,我们立即可以计算泛函  $GD_{(n,F)}$  的第一变分

$$\begin{aligned} \frac{\partial}{\partial t} GD_{(n,F)}(x_i) &= \int_{M_i} F'(S) \frac{\partial}{\partial t}(S) + F(S) (V^i_{,i} - nH^\alpha V^\alpha) dv \\ &= \int_{M_i} F'(S) (2h_{ij}^\alpha V^\alpha_{,ij} + S_{,i} V^i + 2S_{\alpha\alpha\beta} V^\beta - 2h_{ij}^\alpha \bar{R}_{ij\alpha}^\beta V^\beta) \\ &\quad + F(S) (V^i_{,i} - nH^\alpha V^\alpha) dv \\ &= \int_M (2F'(S) h_{ij}^\alpha)_{,ij} V^\alpha + F'(S) S_{,i} V^i + 2F'(S) S_{\alpha\alpha\beta} V^\beta \\ &\quad - 2F'(S) h_{ij}^\alpha \bar{R}_{ij\alpha}^\beta V^\beta - F'(S) S_{,i} V^i - nH^\alpha F(S) V^\alpha \\ &= \int_M [(2F'(S) h_{ij}^\alpha)_{,ij} + 2F'(S) S_{\alpha\alpha\beta}] V^\beta \end{aligned}$$

$$-2F'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}-nF(S)H^{\alpha})V^{\alpha}dv.$$

对于  $GD_{(n, F, \epsilon)}$  泛函, 同理计算可得

$$\begin{aligned}\frac{\partial}{\partial t}GD_{(n, F, \epsilon)}(x_t) &= \int_{M_t} F'(S+\epsilon) \frac{\partial}{\partial t}(S)dv + F(S+\epsilon) \frac{\partial}{\partial t} dv \\ &= \int_{M_t} F'(S+\epsilon) [2h_{ij}^{\alpha}V_{,ij}^{\alpha} + S_{,i}V^i + 2S_{\alpha\beta\beta}V^{\alpha} - 2h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}V^{\alpha}] \\ &\quad + F(S+\epsilon) \left( \sum_i V_{,i}^i - \sum_{\alpha} nH^{\alpha}V^{\alpha} \right) dv \\ &= \int_{M_t} [(2F'(S+\epsilon)h_{ij}^{\alpha})_{,ij}V^{\alpha} + 2F'(S+\epsilon)S_{\alpha\beta\beta}V^{\alpha} \\ &\quad - 2F'(S+\epsilon)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}V^{\alpha} - nF(S+\epsilon)H^{\alpha}V^{\alpha}] dv.\end{aligned}$$

因此我们证明了

**定理 7.1:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, F)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2F'(S)h_{ij}^{\alpha})_{,ij} + (2F'(S)S_{\alpha\beta\beta} - 2F'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta} - nF(S)H^{\alpha}) = 0. \quad (7-3)$$

**定理 7.2:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 有

$$(2F'(S+\epsilon)h_{ij}^{\alpha})_{,ij} + 2F'(S+\epsilon)S_{\alpha\beta\beta} - 2F'(S+\epsilon)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta} - nF(S+\epsilon)H^{\alpha} = 0. \quad (7-4)$$

**定理 7.3:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, F)}$  超曲面当且仅当  $(2F'(S)h_{ij}^{\alpha})_{,ij} + 2F'(S)P_3 + 2F'(S)h_{ij}^{\beta}\bar{R}_{i(n+1)(n+1)j} - nF(S)H = 0$ .

$$(7-5)$$

**定理 7.4:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  超曲面当且仅当  $(2F'(S+\epsilon)h_{ij}^{\alpha})_{,ij} + 2F'(S+\epsilon)P_3 + 2F'(S+\epsilon)h_{ij}^{\beta}\bar{R}_{i(n+1)(n+1)j} - nF(S+\epsilon)H = 0$ .

$$(7-6)$$

**定理 7.5:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^{\alpha} = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, F)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2F'(S)S_{\alpha\beta\beta} - 2F'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta} - nF(S)H^{\alpha} = 0. \quad (7-7)$$

**定理 7.6:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^{\alpha} = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2F'(S+\epsilon)S_{\alpha\beta\beta} - 2F'(S+\epsilon)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta} - nF(S+\epsilon)H^{\alpha} = 0. \quad (7-8)$$

**定理 7.7:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij}^{\alpha} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, F)}$  超曲面当且仅当

$$2F'(S)P_3 + 2F'(S)h_{ij}^{\beta}\bar{R}_{i(n+1)(n+1)j} - nF(S)H = 0. \quad (7-9)$$

**定理 7.8:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij}^{\alpha} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  超曲面当且仅当

$$2F'(S+\epsilon)P_3+2F'(S+\epsilon)h_{ij}\bar{R}_{i(n+1)(n+1)j}-nF(S+\epsilon)H=0. \quad (7-10)$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^{\beta} = -c\delta_{ij}\delta_{\alpha\beta},$$

$$\bar{R}_{AB}^{\gamma} = \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c\sum_i \delta_{Ai}\delta_{iB},$$

$$\bar{R}_{AB}^{\perp} = \sum_{\alpha} \bar{R}_{A\alpha\alpha B} = \sum_{\alpha} -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c\sum_{\alpha} \delta_{A\alpha}\delta_{B\alpha},$$

$$\bar{R}_{\alpha\beta}^{\gamma} = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.$$

于是上面的定理在空间形式之中可以归结于

**定理 7.9:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n, F)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2F'(S)h_{ij}^{\alpha})_{,ij} + 2F'(S)S_{\alpha\alpha\beta\beta} + 2ncF'(S)H^{\alpha} - nF(S)H^{\alpha} = 0. \quad (7-11)$$

**定理 7.10:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2F'(S+\epsilon)h_{ij}^{\alpha}\epsilon)_{,ij} + 2F'(S+\epsilon)S_{\alpha\alpha\beta\beta} + 2ncF'(S+\epsilon)H^{\alpha} - nF(S+\epsilon)H^{\alpha} = 0. \quad (7-12)$$

**定理 7.11:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n, F)}$  超曲面当且仅当

$$(2F'(S)h_{ij})_{,ij} + 2F'(S)P_3 + 2ncF'(S)H - nF(S)H = 0. \quad (7-13)$$

**定理 7.12:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  超曲面当且仅当

$$(2F'(S+\epsilon)h_{ij}\epsilon)_{,ij} + 2F'(S+\epsilon)P_3 + 2ncF'(S+\epsilon)H - nF(S+\epsilon)H = 0. \quad (7-14)$$

**定理 7.13:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^{\alpha} = \text{const}, \forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, F)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2F'(S)S_{\alpha\alpha\beta\beta} + 2ncF'(S)H^{\alpha} - nF(S)H^{\alpha} = 0. \quad (7-15)$$

**定理 7.14:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^{\alpha} = \text{const}, \forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2F'(S+\epsilon)S_{\alpha\alpha\beta\beta} + 2ncF'(S+\epsilon)H^{\alpha} - nF(S+\epsilon)H^{\alpha} = 0. \quad (7-16)$$

**定理 7.15:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}, \forall i, j$ , 那么  $M$  是一个  $GD_{(n, F)}$  超曲面当且仅当

$$2F'(S)P_3 + 2ncF'(S)H - nF(S)H = 0. \quad (7-17)$$

**定理 7.16:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}, \forall i, j$ ,

那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  超曲面当且仅当

$$2F'(S + \epsilon)P_3 + 2ncF'(S + \epsilon)H - nF(S + \epsilon)H = 0. \quad (7-18)$$

## 7.2 $GD_{(n, r)}$ 泛函的第一变分公式

我们知道当  $F(u) = u^r$  时, 泛函  $GD_{n, F}$  和  $GD_{(n, F, \epsilon)}$  分别变为

$$GD_{(n, r)} = \int_M S^r dv, \quad GD_{(n, r, \epsilon)} = \int_M (S + \epsilon)^r dv$$

针对此类重要的特殊情形的计算, 直接利用 7.1 节的定理得到, 有

**定理 7.17:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, r)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2rS^{r-1}h_{ij}^\alpha)_{,ij} + 2rS^{r-1}S_{\alpha\beta\beta} - 2rS^{r-1}h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - nS^r H^\alpha = 0. \quad (7-19)$$

**定理 7.18:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, r, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$\begin{aligned} & (2r(S + \epsilon)^{r-1}h_{ij}^\alpha)_{,ij} + 2r(S + \epsilon)^{r-1}S_{\alpha\beta\beta} \\ & - 2r(S + \epsilon)^{r-1}h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - n(S + \epsilon)^r H^\alpha = 0. \end{aligned} \quad (7-20)$$

**定理 7.19:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, r)}$  超曲面当且仅当

$$(2rS^{r-1}h_{ij})_{,ij} + 2rS^{r-1}P_3 + 2rS^{r-1}h_{ij}\bar{R}_{i(n+1)(n+1)j} - nS^r H = 0. \quad (7-21)$$

**定理 7.20:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, r, \epsilon)}$  超曲面当且仅当

$$\begin{aligned} & (2r(S + \epsilon)^{r-1}h_{ij})_{,ij} + 2r(S + \epsilon)^{r-1}P_3 \\ & + 2r(S + \epsilon)^{r-1}h_{ij}\bar{R}_{i(n+1)(n+1)j} - n(S + \epsilon)^r H = 0. \end{aligned} \quad (7-22)$$

**定理 7.21:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, r)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2rS^{r-1}S_{\alpha\beta\beta} - 2rS^{r-1}h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - nS^r H^\alpha = 0. \quad (7-23)$$

**定理 7.22:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, r, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2r(S + \epsilon)^{r-1}S_{\alpha\beta\beta} - 2r(S + \epsilon)^{r-1}h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - n(S + \epsilon)^r H^\alpha = 0. \quad (7-24)$$

**定理 7.23:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, r)}$  超曲面当且仅当

$$2rS^{r-1}P_3 + 2rS^{r-1}h_{ij}\bar{R}_{i(n+1)(n+1)j} - nS^r H = 0. \quad (7-25)$$

**定理 7.24:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, r, \epsilon)}$  超曲面当且仅当

$$2r(S + \epsilon)^{r-1}P_3 + 2r(S + \epsilon)^{r-1}h_{ij}\bar{R}_{i(n+1)(n+1)j} - n(S + \epsilon)^r H = 0. \quad (7-26)$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^{\beta} = -c\delta_{ij}\delta_{\alpha\beta},$$

$$\bar{R}_{AB}^{\gamma} = \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c\sum_i \delta_{Ai}\delta_{iB},$$

$$\bar{R}_{AB}^{\perp} = \sum_{\alpha} \bar{R}_{A\alpha\alpha B} = \sum_{\alpha} -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c\sum_{\alpha} \delta_{A\alpha}\delta_{B\alpha},$$

$$\bar{R}_{\alpha\beta}^{\gamma} = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.$$

于是上面的定理在空间形式之中可以归结于

**定理 7.25:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n,r)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2rS^{r-1}h_{ij}^{\alpha})_{,ij} + 2rS^{r-1}S_{\alpha\beta\beta} + 2ncrS^{r-1}H^{\alpha} - nS^rH^{\alpha} = 0. \quad (7-27)$$

**定理 7.26:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n,r,\epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2r(S+\epsilon)^{r-1}h_{ij}^{\alpha})_{,ij} + 2r(S+\epsilon)^{r-1}S_{\alpha\beta\beta} + 2ncr(S+\epsilon)^{r-1}H^{\alpha} - n(S+\epsilon)^rH^{\alpha} = 0. \quad (7-28)$$

**定理 7.27:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n,r)}$  超曲面当且仅当

$$(2rS^{r-1}h_{ij})_{,ij} + 2rS^{r-1}P_3 + 2ncrS^{r-1}H - nS^rH = 0. \quad (7-29)$$

**定理 7.28:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n,r,\epsilon)}$  超曲面当且仅当

$$(2r(S+\epsilon)^{r-1}h_{ij})_{,ij} + 2r(S+\epsilon)^{r-1}P_3 + 2ncr(S+\epsilon)^{r-1}H - n(S+\epsilon)^rH = 0. \quad (7-30)$$

**定理 7.29:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^{\alpha} = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n,r)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2rS^{r-1}S_{\alpha\beta\beta} + 2ncrS^{r-1}H^{\alpha} - nS^rH^{\alpha} = 0. \quad (7-31)$$

**定理 7.30:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^{\alpha} = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n,r,\epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2r(S+\epsilon)^{r-1}S_{\alpha\beta\beta} + 2ncr(S+\epsilon)^{r-1}H^{\alpha} - n(S+\epsilon)^rH^{\alpha} = 0. \quad (7-32)$$

**定理 7.31:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n,r)}$  超曲面当且仅当

$$2rS^{r-1}P_3 + 2ncrS^{r-1}H - nS^rH = 0. \quad (7-33)$$

**定理 7.32:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n,r,\epsilon)}$  超曲面当且仅当

$$2r(S+\epsilon)^{r-1}P_3 + 2ncr(S+\epsilon)^{r-1}H - n(S+\epsilon)^rH = 0. \quad (7-34)$$

### 7.3 $GD_{(n, E)}$ 泛函的第一变分公式

我们知道当  $F(u) = e^u$  时, 泛函  $GD_{n, E}$  和  $GD_{(n, E, \epsilon)}$  分别变为

$$GD_{(n, E)} = \int_M e^S dv, \quad GD_{(n, E, \epsilon)} = \int_M e^{S+\epsilon} dv$$

显然  $GD_{(n, E)}$  和  $GD_{(n, E, \epsilon)}$  是相同的在相差一个正系数的意义下, 因此针对此类重要的特殊情形的计算, 直接利用 7.1 节的定理, 有

**定理 7.33:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, E)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2e^S h_{ij}^\alpha)_{,ij} + 2e^S S_{\alpha\beta\beta} - 2e^S h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - ne^S H^\alpha = 0. \quad (7-35)$$

**定理 7.34:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, E)}$  超曲面当且仅当

$$(2e^S h_{ij})_{,ij} + 2e^S P_3 + 2e^S h_{ij} \bar{R}_{i(n+1)(n+1)j} - ne^S H = 0. \quad (7-36)$$

**定理 7.35:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^\alpha = \text{const}, \forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, E)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2S_{\alpha\beta\beta} - 2h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - nH^\alpha = 0. \quad (7-37)$$

**定理 7.36:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij} = \text{const}, \forall i, j$ , 那么  $M$  是一个  $GD_{(n, E)}$  超曲面当且仅当

$$2P_3 + 2h_{ij} \bar{R}_{i(n+1)(n+1)j} - nH = 0. \quad (7-38)$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^\beta = -c\delta_{ij}\delta_{\alpha\beta},$$

$$\bar{R}_{AB}^\tau = \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c \sum_i \delta_{Ai}\delta_{iB},$$

$$\bar{R}_{AB}^\perp = \sum_\alpha \bar{R}_{A\alpha\alpha B} = \sum_\alpha -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c \sum_\alpha \delta_{A\alpha}\delta_{B\alpha},$$

$$\bar{R}_{\alpha\beta}^\tau = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^\perp = pc\delta_{ij}.$$

于是上面的定理在空间形式之中可以归结于

**定理 7.37:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n, E)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2h_{ij}^\alpha e^S)_{,ij} + 2e^S S_{\alpha\beta\beta} + 2nce^S H^\alpha - ne^S H^\alpha = 0. \quad (7-39)$$

**定理 7.38:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n, E)}$  超曲面当且仅当

$$(2h_{ij} e^S)_{,ij} + 2e^S P_3 + 2nce^S H - ne^S H = 0. \quad (7-40)$$

**定理 7.39:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, E)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2S_{\alpha\beta\beta} + 2ncH^\alpha - nH^\alpha = 0. \quad (7-41)$$

**定理 7.40:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, E)}$  超曲面当且仅当

$$2P_3 + 2ncH - nH = 0. \quad (7-42)$$

## 7.4 $GD_{(n, \log)}$ 泛函的第一变分公式

我们知道当  $F(u) = \log u$ ,  $u > 0$  时, 泛函  $GD_{n, \log}$  和  $GD_{(n, \log, \epsilon)}$  分别变为

$$GD_{(n, \log)} = \int_M \log S dv, \quad GD_{(n, \log, \epsilon)} = \int_M \log(S + \epsilon) dv$$

对于  $GD_{(n, \log)}$  泛函, 我们显然要求其没有脐点。针对此类重要的特殊情形的计算, 直接利用 7.1 节的定理得到, 有

**定理 7.41:** 设  $x: M^n \rightarrow N^{n+p}$  是无测地点子流形, 那么  $M$  是一个  $GD_{(n, \log)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$\left(2 \frac{1}{S} h_{ij}^\alpha\right)_{,ij} + 2 \frac{1}{S} S_{\alpha\beta\beta} - 2 \frac{1}{S} h_{ij}^\beta \bar{R}_{j\alpha}^\beta - n \log(S) H^\alpha = 0. \quad (7-43)$$

**定理 7.42:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, \log, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$\left(2 \frac{1}{S + \epsilon} h_{ij}^\alpha\right)_{,ij} + 2 \frac{1}{S + \epsilon} S_{\alpha\beta\beta} - 2 \frac{1}{S + \epsilon} h_{ij}^\beta \bar{R}_{j\alpha}^\beta - n \log(S + \epsilon) H^\alpha = 0. \quad (7-44)$$

**定理 7.43:** 设  $x: M^n \rightarrow N^{n+1}$  是无测地点超曲面, 那么  $M$  是一个  $GD_{(n, \log)}$  超曲面当且仅当

$$\left(2 \frac{1}{S} h_{ij}\right)_{,ij} + 2 \frac{1}{S} P_3 + 2 \frac{1}{S} h_{ij} \bar{R}_{i(n+1)(n+1)j} - n \log(S) H = 0. \quad (7-45)$$

**定理 7.44:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, \log, \epsilon)}$  超曲面当且仅当

$$\left(2 \frac{1}{S + \epsilon} h_{ij}\right)_{,ij} + 2 \frac{1}{S + \epsilon} P_3 + 2 \frac{1}{S + \epsilon} h_{ij} \bar{R}_{i(n+1)(n+1)j} - n \log(S + \epsilon) H = 0. \quad (7-46)$$

**定理 7.45:** 设  $x: M^n \rightarrow N^{n+p}$  是无测地点子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \log)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2 \frac{1}{S} S_{\alpha\beta\beta} - 2 \frac{1}{S} h_{ij}^\beta \bar{R}_{j\alpha}^\beta - n \log(S) H^\alpha = 0. \quad (7-47)$$

**定理 7.46:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个



$GD_{(n, \log, \epsilon)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2 \frac{1}{S+\epsilon} S_{\alpha\beta\beta} - 2 \frac{1}{S+\epsilon} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} - n \log(S+\epsilon) H^{\alpha} = 0. \quad (7-48)$$

**定理 7.47:** 设  $x: M^n \rightarrow N^{n+1}$  是无测地点超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \log)}$  超曲面当且仅当

$$2 \frac{1}{S} P_3 + 2 \frac{1}{S} h_{ij} \bar{R}_{i(n+1)(n+1)j} - n \log(S) H = 0. \quad (7-49)$$

**定理 7.48:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \log, \epsilon)}$  超曲面当且仅当

$$2 \frac{1}{S+\epsilon} P_3 + 2 \frac{1}{S+\epsilon} h_{ij} \bar{R}_{i(n+1)(n+1)j} - n \log(S+\epsilon) H = 0. \quad (7-50)$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^{\beta} = -c\delta_{ij}\delta_{\alpha\beta},$$

$$\bar{R}_{AB}^{\gamma} = \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c \sum_i \delta_{Ai}\delta_{iB},$$

$$\bar{R}_{AB}^{\perp} = \sum_{\alpha} \bar{R}_{A\alpha\alpha B} = \sum_{\alpha} -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c \sum_{\alpha} \delta_{A\alpha}\delta_{B\alpha},$$

$$\bar{R}_{\alpha\beta}^{\gamma} = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.$$

于是上面的定理在空间形式之中可以归结于

**定理 7.49:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的无测地点子流形, 那么  $M$  是一个  $GD_{(n, \log)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$\left(2 \frac{1}{S} h_{ij}^{\alpha}\right)_{,ij} + 2 \frac{1}{S} S_{\alpha\beta\beta} + 2nc \frac{1}{S} H^{\alpha} - n \log(S) H^{\alpha} = 0. \quad (7-51)$$

**定理 7.50:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n, \log, \epsilon)}$  子流形当且仅当对任意的  $\alpha, (n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$\left(2 \frac{1}{S+\epsilon} h_{ij}^{\alpha}\right)_{,ij} + 2 \frac{1}{S+\epsilon} S_{\alpha\beta\beta} + 2nc \frac{1}{S+\epsilon} H^{\alpha} - n \log(S+\epsilon) H^{\alpha} = 0. \quad (7-52)$$

**定理 7.51:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的无测地点超曲面, 那么  $M$  是一个  $GD_{(n, \log)}$  超曲面当且仅当

$$\left(2 \frac{1}{S} h_{ij}\right)_{,ij} + 2 \frac{1}{S} P_3 + 2nc \frac{1}{S} H - n \log(S) H = 0. \quad (7-53)$$

**定理 7.52:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n, F, \epsilon)}$  超曲面当且仅当

$$\left(2 \frac{1}{S+\epsilon} h_{ij}\right)_{,ij} + 2 \frac{1}{S+\epsilon} P_3 + 2nc \frac{1}{S+\epsilon} H - n \log(S+\epsilon) H = 0. \quad (7-54)$$

**定理 7.53:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的无测地点子流形并且  $h_{ij}^{\alpha} =$

$\text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \log)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2 \frac{1}{S} S_{\alpha\beta\beta} + 2nc \frac{1}{S} H^\alpha - n \log(S) H^\alpha = 0. \quad (7-55)$$

**定理 7.54:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \log, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2 \frac{1}{S+\epsilon} S_{\alpha\beta\beta} + 2nc \frac{1}{S+\epsilon} H^\alpha - n \log(S+\epsilon) H^\alpha = 0. \quad (7-56)$$

**定理 7.55:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的无测地点超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \log)}$  超曲面当且仅当

$$2 \frac{1}{S} P_3 + 2nc \frac{1}{S} H - n \log(S) H = 0. \quad (7-57)$$

**定理 7.56:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \log, \epsilon)}$  超曲面当且仅当

$$2 \frac{1}{S+\epsilon} P_3 + 2nc \frac{1}{S+\epsilon} H - n \log(S+\epsilon) H = 0. \quad (7-58)$$

## 7.5 $GD_{(n, \sin)}$ 泛函的第一变分公式

我们知道当  $F(u) = \sin(S)$  时, 泛函  $GD_{(n, \sin)}$  和  $GD_{(n, \sin, \epsilon)}$  分别变为

$$GD_{(n, \sin)} = \int_M \sin(S) dv, \quad GD_{(n, \sin, \epsilon)} = \int_M \sin(S+\epsilon) dv$$

针对此类重要的特殊情形的计算, 直接利用 7.1 节的定理得到, 有

**定理 7.57:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, \sin)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2\cos(S)h_{ij}^\alpha)_{,ij} + 2\cos(S)S_{\alpha\beta\beta} - 2\cos(S)h_{ij}^\beta \bar{R}_{j\alpha}^\beta - n\sin(S)H^\alpha = 0. \quad (7-59)$$

**定理 7.58:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形, 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2\cos(S+\epsilon)h_{ij}^\alpha)_{,ij} + 2\cos(S+\epsilon)S_{\alpha\beta\beta} - 2\cos(S+\epsilon)h_{ij}^\beta \bar{R}_{j\alpha}^\beta - n\sin(S+\epsilon)H^\alpha = 0. \quad (7-60)$$

**定理 7.59:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, \sin)}$  超曲面当且仅当

$$(2\cos(S)h_{ij})_{,ij} + 2\cos(S)P_3 + 2\cos(S)h_{ij} \bar{R}_{i(n+1)(n+1)j} - n\sin(S)H = 0. \quad (7-61)$$

**定理 7.60:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面, 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  超曲面当且仅当

$$(2\cos(S+\epsilon)h_{ij})_{,ij} + 2\cos(S+\epsilon)P_3 + 2\cos(S+\epsilon)h_{ij}\bar{R}_{i(n+1)(n+1)j} - n\sin(S+\epsilon)H = 0. \quad (7-62)$$

**定理 7.61:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \sin)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2\cos(S)S_{\alpha\beta\beta} - 2\cos(S)h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - n\sin(S)H^\alpha = 0. \quad (7-63)$$

**定理 7.62:** 设  $x: M^n \rightarrow N^{n+p}$  是子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2\cos(S+\epsilon)S_{\alpha\beta\beta} - 2\cos(S+\epsilon)h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - n\sin(S+\epsilon)H^\alpha = 0. \quad (7-64)$$

**定理 7.63:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \sin)}$  超曲面当且仅当

$$2\cos(S)P_3 + 2\cos(S)h_{ij}\bar{R}_{i(n+1)(n+1)j} - n\sin(S)H = 0. \quad (7-65)$$

**定理 7.64:** 设  $x: M^n \rightarrow N^{n+1}$  是超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  超曲面当且仅当

$$2\cos(S+\epsilon)P_3 + 2\cos(S+\epsilon)h_{ij}\bar{R}_{i(n+1)(n+1)j} - n\sin(S+\epsilon)H = 0. \quad (7-66)$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^\beta = -c\delta_{ij}\delta_{\alpha\beta},$$

$$\bar{R}_{AB}^\tau = \sum_i \bar{R}_{Ai\tau B} = \sum_i -c(\delta_{Ai}\delta_{\tau B} - \delta_{AB}\delta_{\tau i}) = nc\delta_{AB} - c \sum_i \delta_{Ai}\delta_{\tau B},$$

$$\bar{R}_{AB}^\perp = \sum_\alpha \bar{R}_{A\alpha\alpha B} = \sum_\alpha -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c \sum_\alpha \delta_{A\alpha}\delta_{B\alpha},$$

$$\bar{R}_{\alpha\beta}^\tau = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^\perp = pc\delta_{ij}.$$

于是上面的定理在空间形式之中可以归结于

**定理 7.65:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n, \sin)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2\cos(S)h_{ij}^\alpha)_{,ij} + 2\cos(S)S_{\alpha\beta\beta} + 2nc\cos(S)H^\alpha - n\sin(S)H^\alpha = 0. \quad (7-67)$$

**定理 7.66:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形, 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$(2\cos(S+\epsilon)h_{ij}^\alpha)_{,ij} + 2\cos(S+\epsilon)S_{\alpha\beta\beta} + 2nc\cos(S+\epsilon)H^\alpha - n\sin(S+\epsilon)H^\alpha = 0. \quad (7-68)$$

**定理 7.67:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个  $GD_{(n, \sin)}$  超曲面当且仅当

$$(2\cos(S)h_{ij})_{,ij} + 2\cos(S)P_3 + 2nc\cos(S)H - n\sin(S)H = 0. \quad (7-69)$$

**定理 7.68:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面, 那么  $M$  是一个

$GD_{(n, \sin, \epsilon)}$  超曲面当且仅当

$$(2\cos(S + \epsilon)h_{ij})_{,ij} + 2\cos(S + \epsilon)P_3 + 2nccos(S + \epsilon)H - n\sin(S + \epsilon)H = 0. \quad (7-70)$$

**定理 7.69:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \sin)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2\cos(S)S_{\alpha\beta\beta} + 2nccos(S)H^\alpha - n\sin(S)H^\alpha = 0. \quad (7-71)$$

**定理 7.70:** 设  $x: M \rightarrow R^{n+p}(c)$  是空间形式中的子流形并且  $h_{ij}^\alpha = \text{const}$ ,  $\forall i, j, \alpha$ , 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  子流形当且仅当对任意的  $\alpha$ ,  $(n+1) \leq \alpha \leq (n+p)$ , 下式成立

$$2\cos(S + \epsilon)S_{\alpha\beta\beta} + 2nccos(S + \epsilon)H^\alpha - n\sin(S + \epsilon)H^\alpha = 0. \quad (7-72)$$

**定理 7.71:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \sin)}$  超曲面当且仅当

$$2\cos(S)P_3 + 2nccos(S)H - n\sin(S)H = 0. \quad (7-73)$$

**定理 7.72:** 设  $x: M \rightarrow R^{n+1}(c)$  是空间形式中的超曲面并且  $h_{ij} = \text{const}$ ,  $\forall i, j$ , 那么  $M$  是一个  $GD_{(n, \sin, \epsilon)}$  超曲面当且仅当

$$2\cos(S + \epsilon)P_3 + 2nccos(S + \epsilon)H - n\sin(S + \epsilon)H = 0. \quad (7-74)$$

## 第 8 章 单位球面中临界子流形的例子

第 7 章我们计算了各类曲率模长泛函的第一变分公式, 根据第一变分公式, 我们可以构造各种  $GD_{(n, F)}$  子流形的例子, 特别的, 对于具体的函数形式  $F$  可以得到更加丰富的结果。

### 8.1 $GD_{(n, F)}$ 子流形的例子

在本节, 我们给出多种  $GD_{(n, F)}$  子流形的例子。这些例子在间隙现象的讨论时很有用处。

特别的, 我们关注单位球面  $S^{n+1}(1)$  之中的  $GD_{(n, F)}$  并且等参的超曲面。我们知道, 单位球面之中的等参超曲面的所有主曲率为

$$\{k_1, k_2, \dots, k_i, \dots, k_n\} = \text{const}$$

那么

$$P_1 = nH = \text{const}, P_2 = S = \text{const}, P_3 = \text{const}.$$

因此, 单位曲面之中的  $GD_{(n, F)}$  等参超曲面方程变为

$$2F'(P_2)P_3 + 2F'(P_2)P_1 - F(P_2)P_1 = 0.$$

**例 8.1:** 全测地超曲面按照其定义, 我们知道所有的主曲率为

$$k_1 = k_2 = \dots = 0.$$

于是, 可以计算为

$$P_1 = 0, P_2 = 0, P_3 = 0.$$

代入上面的方程, 我们可以做结论为对于任意的参数函数  $F \in C^3[0, \infty)$ , 全测地超曲面  $M$  为  $GD_{(n, F)}$  超曲面。

**例 8.2:** 全脐非全测地的超曲面, 按照定义, 我们可知所有的主曲率为

$$k_1 = k_2 = \dots = k_n = \lambda \neq 0.$$

各种曲率函数的计算为

$$P_1 = n\lambda, P_2 = n\lambda^2, P_3 = n\lambda^3.$$

代入方程, 可得全脐非全测地超曲面对于满足条件

$$2F'(n\lambda^2)\lambda^2 + 2F'(n\lambda^2) - F(n\lambda^2) = 0$$

的函数  $F \in C^3(0, \infty)$  都是  $GD_{(n, F)}$  超曲面。显然下面的函数是满足以上条件的

$$F(u) = F(u_0) \left( \frac{u+n}{u_0+n} \right)^{\frac{n}{2}}$$

**例 8.3:** 对于维数为偶数  $n \equiv 0 \pmod{2}$  的特殊 Clifford 超曲面

$$C_{\frac{n}{2}, \frac{n}{2}} = S^{\frac{n}{2}} \left( \frac{1}{\sqrt{2}} \right) \times S^{\frac{n}{2}} \left( \frac{1}{\sqrt{2}} \right) \rightarrow S^{n+1}(1).$$

我们知道所有的主曲率为

$$k_1 = \cdots = k_{\frac{n}{2}} = 1, \quad k_{\frac{n}{2}+1} = \cdots = k_n = -1.$$

于是可以计算所有的曲率函数  $P_1, P_2, P_3$  为

$$P_1 = 0, \quad P_2 = n, \quad P_3 = 0.$$

于是我们得到  $C_{\frac{n}{2}, \frac{n}{2}}$  对于任何函数  $F \in C^3(0, \infty)$  都是  $GD_{(n, F)}$  超曲面。

**例 8.4:** 对于单位球面之中的具有两个不同主曲率的超曲面, 我们有

$$\lambda, \mu, \quad 0 < \lambda, \mu < 1, \quad \lambda^2 + \mu^2 = 1,$$

$$S^m(\lambda) \times S^{n-m}(\mu) \rightarrow S^{n+1}(1), \quad 1 \leq m \leq n-1.$$

我们需要在上面的超曲面之中决定出所有的  $GD_{(n, F)}$  超曲面。显然的, 通过计算, 我们有

$$k_1 = k_2 = \cdots = k_m = \frac{\mu}{\lambda}, \quad k_{m+1} = k_{m+2} = \cdots = k_n = -\frac{\lambda}{\mu}.$$

于是, 曲率函数  $P_1 = nH, P_2 = S, P_3$  分别为

$$P_1 = m \frac{\mu}{\lambda} - (n-m) \frac{\lambda}{\mu},$$

$$P_2 = m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2},$$

$$P_3 = m \frac{\mu^3}{\lambda^3} - (n-m) \frac{\lambda^3}{\mu^3}.$$

假设  $\frac{\mu}{\lambda} = x > 0$ , 于是  $GD_{(n, F)}$  超曲面方程变为

$$2F' \left[ mx^2 + (n-m) \frac{1}{x^2} \right] \left[ mx^6 - (n-m) \right] +$$

$$\left\{ 2F' \left[ mx^2 + (n-m) \frac{1}{x^2} \right] - F \left[ mx^2 + (n-m) \frac{1}{x^2} \right] \right\} \left[ mx^4 - (n-m)x^2 \right] = 0.$$

对于具体的函数, 通过求解具体的代数方程, 可以构造出临界超曲面。

**例 8.5:** 当  $F(S) = 1$  时, 具有两个不同主曲率的  $GD_{(n, F)}$  等参超曲面即为极小等参超曲面, 即是 Clifford Torus。

$$C_{m, n-m} = S^m \left( \sqrt{\frac{m}{n}} \right) \times S^{n-m} \left( \sqrt{\frac{n-m}{n}} \right), \quad 1 \leq m \leq n-1.$$

而且满足  $P_1 \equiv 0, P_2 \equiv n, P_3 = (n-m)\sqrt{\frac{n-m}{m}} - m\sqrt{\frac{m}{n-m}}$ 。我们假设  $F_1$  是另外一个函数满足  $F_1 \in C^3(0, +\infty)$ 。如果某个  $C_{m, n-m}$  同时也是  $GD_{(n, F_1)}$  超曲面, 那么必须满足  $F_1'(n) \left[ \sqrt{\frac{(n-m)^3}{m}} - \sqrt{\frac{m^3}{n-m}} \right] = 0$ 。因此我们可以做一些结论: 如果  $F_1'(n) = 0$ , 那么所有的  $C_{m, n-m}$  都是  $GD_{(n, F_1)}$  超曲面; 如果  $F_1'(n) \neq 0$ , 那么某个  $C_{m, n-m}$  是  $GD_{(n, F_1)}$  超曲面当且仅当

$$n \equiv 0 \pmod{2}, m = \frac{n}{2}, C_{m, n-m} = C_{\frac{n}{2}, \frac{n}{2}}.$$

**例 8.6:** 对于单位球面之中的具有两个不同的主曲率的超曲面, 我们寻求满足  $S = n$  的所有 Torus。我们知道

$$\begin{aligned} \lambda, \mu, 0 < \lambda, \mu < 1, \lambda^2 + \mu^2 &= 1, \\ S^m(\lambda) \times S^{n-m}(\mu) &\rightarrow S^{n+1}(1), 1 \leq m \leq n-1. \end{aligned}$$

显然的, 所有的主曲率为

$$k_1 = k_2 = \cdots = k_m = \frac{\mu}{\lambda}, k_{m+1} = k_{m+2} = \cdots = k_n = -\frac{\lambda}{\mu}.$$

于是曲率函数  $S$  为

$$S = m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2}.$$

假设  $\frac{\mu}{\lambda} = x > 0$ , 于是

$$S = mx^2 + (n-m) \frac{1}{x^2}.$$

如果  $S = n$ , 我们有方程

$$n = mx^2 + (n-m) \frac{1}{x^2}.$$

解这个方程得到

$$x_1 = \sqrt{\frac{n-m}{m}}, x_2 = 1, \forall m \in N, 1 \leq m \leq n-1.$$

所以

$$C_{m, n-m}: S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right) \rightarrow S^{n+1}(1), 1 \leq m \leq n-1$$

和

$$S^m\left(\sqrt{\frac{1}{2}}\right) \times S^{n-m}\left(\sqrt{\frac{1}{2}}\right) \rightarrow S^{n+1}(1), 1 \leq m \leq n-1$$

是满足  $\rho = n$  的所有 Torus。

上面我们研究了超曲面的情形, 下面我们研究子流形的情形, 下面的子流形是微分几何之中的著名的例子, 称为 Veronese 曲面。我们需要利用高维情形的  $GD_{(n, F)}$  子流形 Euler - Lagrange 公式

$$2F'(S)S_{\alpha\beta\beta} + 2nF'(S)H^\alpha - nF(S)H^\alpha = 0.$$

**例 8.7:** 假设  $(x, y, z)$  是三维欧式空间  $R^3$  的自然标架, 假设  $(u_1, u_2, u_3, u_4, u_5)$  是五维欧式空间  $R^5$  的自然标架, 我们定义如下的映射

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{3}}yz, \quad u_2 = \frac{1}{\sqrt{3}}xz, \quad u_3 = \frac{1}{\sqrt{3}}xy \\ u_4 &= \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2) \\ x^2 + y^2 + z^2 &= 3 \end{aligned}$$

这个映射决定了一个等距嵌入  $x: RP^2 = S^2(\sqrt{3})/Z_2 \rightarrow S^4(1)$ , 我们称其为 Veronese 曲面, 通过简单的计算, 我们知道第二基本型为

$$A_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

通过上面的第二基本型和定义, 我们可以计算得到

$$H^3 = H^4 = 0, \quad S_{33} = S_{44} = \frac{2}{3}, \quad S = \rho = \frac{4}{3},$$

$$S_{34} = S_{43} = 0, \quad S_{333} = S_{344} = S_{433} = S_{444} = 0.$$

显然的 Veronese 曲面对于任意的函数  $F \in C^3(0, \infty)$  都是  $GD_{(2, F)}$  曲面。

## 8.2 $GD_{(n, r)}$ 子流形的例子

在本节我们给出  $GD_{(n, r)}$  子流形的例子。对于单位球面  $S^{n+1}(1)$  之中的等参超曲面, 我们知道所有的主曲率  $\{k_1, k_2, \dots, k_i, \dots, k_n\}$  都是常数, 显然曲率  $P_1, P_2, P_3$  也都是常数。我们知道单位球面之中的等参超曲面是  $GD_{(n, r)}$  超曲面当且仅当满足方程

$$S^{r-1}(2rP_3 + 2rP_1 - SP_1) = 0.$$

**例 8.8:** 全测地超曲面是  $GD_{(n, r)}$  超曲面。此时要求对参数  $r$  的取值为  $r = 1, 2, \dots, [3, \infty)$ 。实际上, 全测地超曲面意味着所有主曲率都为 0, 因此, 曲率  $P_1, P_2, P_3$  都为 0, 所以方程自然满足。

**例 8.9:** 全脐非全测地超曲面的定义为, 所有主曲率相等为常数而且不等于 0。即是



$$k_1 = k_2 = \cdots = k_n = \lambda \neq 0.$$

经过简单的计算, 我们可得

$$P_1 = n\lambda, P_2 = S = n\lambda^2, P_3 = n\lambda^3.$$

代入方程可得

$$(n - 2r)\lambda^2 = 2r.$$

即是  $\lambda$  必须满足上面的方程才是  $GD_{(n, r)}$  子流形。所以  $r$  必须满足  $0 < r < \frac{n}{2}$ .

**例 8.10:** 对于单位球面之中的一个维数为偶数  $n \equiv 0 \pmod{2}$  的特殊子流形

$$C_{\frac{n}{2}, \frac{n}{2}} = S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \times S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \rightarrow S^{n+1}(1).$$

经过简单的计算我们知道所有的主曲率为

$$k_1 = \cdots = k_{\frac{n}{2}} = 1, k_{\frac{n}{2}+1} = \cdots = k_n = -1.$$

那么对于  $P_1, P_2, P_3$  我们可以计算得到

$$P_1 = 0, P_2 = n, P_3 = 0$$

显然  $C_{\frac{n}{2}, \frac{n}{2}}$  不是全测地超曲面, 也没有测地点。代入方程我们可以得到结论: 对于任何参数  $r$ ,  $C_{\frac{n}{2}, \frac{n}{2}}$  是单位球面之中的  $GD_{(n, r)}$  超曲面。

**例 8.11:** 对于单位球面之中的具有两个不同主曲率的超曲面, 我们有

$$\lambda, \mu, 0 < \lambda, \mu < 1, \lambda^2 + \mu^2 = 1, \\ S^m(\lambda) \times S^{n-m}(\mu) \rightarrow S^{n+1}(1), 1 \leq m \leq n-1.$$

我们需要在上面的超曲面之中决定出所有的  $GD_{(n, F)}$  超曲面。显然, 通过计算, 我们有

$$k_1 = \cdots = k_m = \frac{\mu}{\lambda}, k_{m+1} = \cdots = k_n = -\frac{\lambda}{\mu}.$$

于是, 曲率函数  $P_1 = nH, P_2 = S, P_3$  分别为

$$P_1 = m \frac{\mu}{\lambda} - (n-m) \frac{\lambda}{\mu}, \\ P_2 = m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2}, \\ P_3 = m \frac{\mu^3}{\lambda^3} - (n-m) \frac{\lambda^3}{\mu^3}.$$

假设  $\frac{\mu}{\lambda} = x > 0$ , 于是  $GD_{(n, r)}$  超曲面方程变为

$$(2rm - m^2)x^6 + [2rm + m(n-m)]x^4 - \\ [2r(n-m) + m(n-m)]x^2 + (n-m)^2 - 2r(n-m) = 0.$$

对于固定的参数  $(n, r)$ , 需要寻求的解是  $(m, x)$ , 由此可以确定单位球面之中的

所有的具有两个不同主曲率的等参  $GD_{(n,r)}$  超曲面。实际上令  $y = x^2$ , 于是6次方程可以变为

$$(2rm - m^2)y^3 + [2rm + m(n - m)]y^2 - [2r(n - m) + m(n - m)]y + (n - m)^2 - 2r(n - m) = 0.$$

再利用3次代数方程的求解法则, 可以求出解, 过程讨论比较复杂, 但是思想简洁, 留给读者作为一个小问题。

**例 8.12:** 子流形情形。在微分几何之中, 有一个著名的例子被称为 Veronese 曲面。通过前面几节的计算可知 Veronese 曲面是  $GD_{(n,r)}$  曲面, 此处  $r$  的取值为  $r \in \mathbb{R}$ 。

### 8.3 $GD_{(n,E)}$ 子流形的例子

在本节我们研究  $GD_{(n,E)}$  子流形, 首先我们考虑超曲面的情形。对于单位球面  $S^{n+1}(1)$  之中的等参超曲面, 我们根据等参超曲面的定义知道  $\{k_1, k_2, \dots, k_i, \dots, k_n\} = \text{const.}$  因此, 曲率函数  $P_1, P_2, P_3$  都是常数。于是  $GD_{(n,E)}$  超曲面方程变为

$$2P_3 + P_1 = 0.$$

**例 8.13:** 全测地超曲面是  $GD_{(n,r)}$  超曲面。此时要求对参数  $r$  的取值为  $r = 1, 2, \dots, [3, \infty)$ 。实际上, 全测地超曲面意味着所有主曲率都为0, 因此, 曲率  $H, S, P_3$  都为0, 所以方程自然满足。

**例 8.14:** 全脐非全测地的超曲面, 按照定义, 我们可知所有的主曲率为

$$k_1 = k_2 = \dots = k_n = \lambda \neq 0.$$

各种曲率函数的计算为

$$P_1 = n\lambda, P_2 = n\lambda^2, P_3 = n\lambda^3.$$

代入方程, 可得

$$2\lambda^2 + 1 = 0.$$

显然是不满足的, 故所有的全脐非测地超曲面不是  $GD_{(n,E)}$  超曲面。

**例 8.15:** 对于单位球面之中的一个维数为偶数  $n \equiv 0 \pmod{2}$  的特殊子流形

$$C_{\frac{n}{2}, \frac{n}{2}} = S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \times S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \rightarrow S^{n+1}(1).$$

经过简单的计算我们知道所有的主曲率为

$$k_1 = k_2 = \dots = k_{\frac{n}{2}} = 1, k_{\frac{n}{2}+1} = k_{\frac{n}{2}+2} = \dots = k_n = -1.$$

那么对于  $P_1, P_2, P_3$  我们可以计算得到

$$P_1 = 0, P_2 = n, P_3 = 0.$$

显然  $C_{\frac{n}{2}, \frac{n}{2}}$  不是全脐超曲面, 也没有脐点。代入方程, 我们可以得到结论:  $C_{\frac{n}{2}, \frac{n}{2}}$  是单位球面之中的  $GD_{(n,E)}$  超曲面。

**例 8.16:** 对于单位球面之中的具有两个不同主曲率的超曲面, 我们有

$$\lambda, \mu, 0 < \lambda, \mu < 1, \lambda^2 + \mu^2 = 1,$$

$$S^m(\lambda) \times S^{n-m}(\mu) \rightarrow S^{n+1}(1), 1 \leq m \leq n-1.$$

我们需要在上面的超曲面之中决定出所有的  $GD_{(n, E)}$  超曲面。显然的, 通过计算, 我们有

$$k_1 = k_2 = \cdots = k_m = \frac{\mu}{\lambda}, k_{m+1} = k_{m+2} = \cdots = k_n = -\frac{\lambda}{\mu}.$$

于是, 曲率函数  $P_1 = nH, P_2 = S, P_3$  分别为

$$P_1 = m \frac{\mu}{\lambda} - (n-m) \frac{\lambda}{\mu},$$

$$P_2 = m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2},$$

$$P_3 = m \frac{\mu^3}{\lambda^3} - (n-m) \frac{\lambda^3}{\mu^3}.$$

假设  $\frac{\mu}{\lambda} = x > 0$ , 于是  $GD_{(n, E)}$  超曲面方程变为

$$2mx^6 + mx^4 - (n-m)x^2 - 2(n-m) = 0.$$

通过求解上面的代数方程, 可以构造出临界超曲面。实际上, 令  $y = x^2$ , 则上面的 6 次代数方程变为

$$2my^3 + my^2 - (n-m)y - 2(n-m) = 0.$$

在利用 3 次代数方程的求解法则, 可以求出解, 过程讨论比较复杂, 但是思想简洁, 留给读者作为一个小问题。

**例 8.17:** 经典的 Clifford Torus.

$$C_{m, n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), 1 \leq m \leq n-1.$$

是极小子流形满足  $H \equiv 0$ 。如果某个  $C_{m, n-m}$  是  $GD_{(n, E)}$  超曲面, 那么必须有

$$n \equiv 0 \pmod{2}, m = \frac{n}{2}, C_{m, n-m} = C_{\frac{n}{2}, \frac{n}{2}}.$$

**例 8.18:** 子流形情形。在微分几何之中, 有一个著名的例子被称为 Veronese 曲面。通过前面几节的计算可知 Veronese 曲面是  $GD_{(n, E)}$  曲面。

## 8.4 $GD_{(n, \log)}$ 子流形的例子

在本节我们研究单位球面  $S^{n+1}(1)$  之中的无脐点的  $GD_{(n, \log)}$  子流形, 特别的, 我们关注单位曲面之中的等参超曲面, 根据等参超曲面的定义我们知道所有的主曲率满足  $|k_1, k_2, \cdots, k_i, \cdots, k_n| = \text{const}$ , 于是曲率函数变量  $P_1, P_2, P_3$  都为常

数。于是  $GD_{(n, \log)}$  超曲面方程变为

$$2P_3 + 2P_1 - S \log(S) P_1 = 0.$$

**例 8.19:** 全脐非全测地的超曲面, 按照定义, 我们可知所有的主曲率为

$$k_1 = k_2 = \cdots = k_n = \lambda \neq 0.$$

各种曲率函数的计算为

$$P_1 = n\lambda, P_2 = n\lambda^2, P_3 = n\lambda^3.$$

代入方程, 可得

$$2\lambda^2 + 2 - n\lambda^2 \log(n\lambda^2) = 0.$$

所以  $\lambda$  必须满足上面的等式, 全脐非全测地超曲面才是  $GD_{(n, \log)}$  超曲面。

**例 8.20:** 对于如下的一个特殊超曲面, 维数满足  $n \equiv 0 \pmod{2}$

$$C_{\frac{n}{2}, \frac{n}{2}} = S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \times S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \rightarrow S^{n+1}(1).$$

我们知道所有的主曲率为

$$k_1 = k_2 = \cdots = k_{\frac{n}{2}} = 1, k_{\frac{n}{2}+1} = k_{\frac{n}{2}+2} = \cdots = k_n = -1.$$

于是我们可以做计算曲率函数  $P_1, P_2, P_3$  分别为

$$P_1 = 0, P_2 = n, P_3 = 0.$$

于是我们可以结论为  $C_{\frac{n}{2}, \frac{n}{2}}$  是单位球面  $S^{n+1}(1)$  之中的  $GD_{(n, \log)}$  超曲面。

**例 8.21:** 对于单位球面之中的具有两个不同主曲率的超曲面, 我们有

$$\lambda, \mu, 0 < \lambda, \mu < 1, \lambda^2 + \mu^2 = 1,$$

$$S^m(\lambda) \times S^{n-m}(\mu) \rightarrow S^{n+1}(1), 1 \leq m \leq n-1.$$

我们需要在上面的超曲面之中决定出所有的  $GD_{(n, \log)}$  超曲面。显然, 通过计算,

我们有  $k_1 = k_2 = \cdots = k_m = \frac{\mu}{\lambda}, k_{m+1} = k_{m+2} = \cdots = k_n = -\frac{\lambda}{\mu}.$

于是, 曲率函数  $P_1 = nH, P_2 = S, P_3$  分别为

$$P_1 = m \frac{\mu}{\lambda} - (n-m) \frac{\lambda}{\mu},$$

$$P_2 = m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2},$$

$$P_3 = m \frac{\mu^3}{\lambda^3} - (n-m) \frac{\lambda^3}{\mu^3}.$$

假设  $\frac{\mu}{\lambda} = x > 0$ , 于是  $GD_{(n, \log)}$  超曲面方程变为

$$2 \frac{mx^6 + mx^4 - (n-m)x^2 - (n-m)}{[mx^2 + (n-m)\frac{1}{x^2}]} -$$

$$\log(mx^2 + (n-m)\frac{1}{x^2})[mx^4 - (n-m)x^2] = 0.$$

通过求解函数方程, 可以构造出临界超曲面。

**例 8.22:** 经典的 Clifford Torus

$$C_{m, n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), 1 \leq m \leq n-1.$$

是极小子流形具有  $H=0$ ,  $S=n$ 。如果某个  $C_{m, n-m}$  是  $GD_{(n, \log)}$  超曲面, 我们可以得到结论为

$$n \equiv 0 \pmod{2}, m = \frac{n}{2}, C_{m, n-m} = C_{\frac{n}{2}, \frac{n}{2}}.$$

**例 8.23:** 子流形情形。在微分几何之中, 有一个著名的例子被称为 Veronese 曲面。通过前面几节的计算可知 Veronese 曲面是  $GD_{(n, \log)}$  曲面。

## 8.5 $GD_{(n, \sin)}$ 子流形的例子

在本节, 我们给出多种  $GD_{(n, \sin)}$  子流形的例子。这些例子在间隙现象的讨论时很有用处。特别的, 我们关注单位球面  $S^{n+1}(1)$  之中的  $GD_{(n, \sin)}$  等参的超曲面。我们知道, 单位球面之中的等参超曲面的所有主曲率为  $\{k_1, \dots, k_i, \dots, k_n\} = \text{const}$ , 那么  $P_1, P_2, P_3$  都为常数。因此,  $GD_{(n, \sin)}$  超曲面方程变为

$$2\cos(S)P_3 + 2\cos(S)P_1 - \sin(S)P_1 = 0.$$

**例 8.24:** 全测地超曲面按照其定义, 我们知道所有的主曲率为

$$k_1 = k_2 = \dots = 0$$

于是, 可以计算为

$$P_1 = 0, P_2 = 0, P_3 = 0.$$

代入上面的方程, 可知全测地超曲面  $M$  为  $GD_{(n, \sin)}$  超曲面。

**例 8.25:** 全脐非全测地的超曲面, 按照定义, 我们可知所有的主曲率为

$$k_1 = k_2 = \dots = k_n = \lambda \neq 0.$$

各种曲率函数的计算为

$$P_1 = n\lambda, P_2 = n\lambda^2, P_3 = n\lambda^3.$$

代入方程, 可得

$$2\cos(n\lambda^2)\lambda^2 + 2\cos(n\lambda^2) - \sin(n\lambda^2) = 0.$$

所以  $\lambda$  必须满足上面的等式, 全脐全非测地超曲面才是  $GD_{(n, \sin)}$  超曲面。

**例 8.26:** 对于维数为偶数  $n \equiv 0 \pmod{2}$  的特殊超曲面

$$C_{\frac{n}{2}, \frac{n}{2}} = S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \times S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \rightarrow S^{n+1}(1).$$

我们知道所有的主曲率为

$$k_1 = k_2 = \dots = k_{\frac{n}{2}} = 1, k_{\frac{n}{2}+1} = k_{\frac{n}{2}+2} = \dots = k_n = -1.$$

于是可以计算所有的曲率函数  $P_1, P_2, P_3$  为

$$P_1 = 0, P_2 = n, P_3 = 0.$$

于是我们得到  $C_{\frac{n}{2}, \frac{n}{2}}$  是  $GD_{(n, \sin)}$  超曲面。

**例 8.27:** 对于单位球面之中的具有两个不同主曲率的超曲面, 我们有

$$\lambda, \mu, 0 < \lambda, \mu < 1, \lambda^2 + \mu^2 = 1,$$

$$S^m(\lambda) \times S^{n-m}(\mu) \rightarrow S^{n+1}(1), 1 \leq m \leq n-1.$$

我们需要在上面的超曲面之中决定出所有的  $GD_{(n, \sin)}$  超曲面。显然, 通过计算, 我们有

$$k_1 = k_2 = \cdots = k_m = \frac{\mu}{\lambda}, k_{m+1} = k_{m+2} = \cdots = k_n = -\frac{\lambda}{\mu}.$$

于是, 曲率函数  $P_1 = nH, P_2 = S, P_3$  分别为

$$P_1 = m \frac{\mu}{\lambda} - (n-m) \frac{\lambda}{\mu},$$

$$P_2 = m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2},$$

$$P_3 = m \frac{\mu^3}{\lambda^3} - (n-m) \frac{\lambda^3}{\mu^3}.$$

假设  $\frac{\mu}{\lambda} = x > 0$ , 于是  $GD_{(n, \sin)}$  超曲面方程变为

$$2\cos(mx^2 + (n-m)\frac{1}{x^2})[mx^6 - (n-m)] +$$

$$\left[2\cos(mx^2 + (n-m)\frac{1}{x^2}) - \sin(mx^2 + (n-m)\frac{1}{x^2})\right][mx^4 - (n-m)x^2] = 0.$$

通过求解上面的函数方程, 可以构造出临界超曲面。

**例 8.28:** 经典的 Clifford Torus

$$C_{m, n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), 1 \leq m \leq n-1.$$

是极小子流形具有  $H \equiv 0, S \equiv n$ 。如果某个  $C_{m, n-m}$  是  $GD_{(n, \sin)}$  - Willmore 超曲面, 我们可以做结论为

$$n \equiv 0 \pmod{2}, m = \frac{n}{2}, C_{m, n-m} = C_{\frac{n}{2}, \frac{n}{2}}.$$

**例 8.29:** 子流形情形。在微分几何之中, 有一个著名的例子被称为 Veronese 曲面。通过前面几节的计算可知 Veronese 曲面是  $GD_{(n, \sin)}$  曲面。

## 第9章 第二变分和稳定性

泛函的第二变分是讨论其临界点子流形的稳定性的基础, 而稳定性刻画了泛函的局部极值特征。

### 9.1 $GD_{(n, F)}$ 泛函的第二变分公式

第6章我们计算了  $W_{(n, F)}$  - Willmore 泛函的第一变分, 为了讨论临界点子流形的稳定性, 第二变分的计算是非常必要的。首先需要几个引理。

**引理 9.1:** 设  $x: M \rightarrow N$  是子流形, 协变导数的差异如下

$$\bar{R}_{i\beta j\alpha; p} = \bar{R}_{i\beta\alpha, p} - \sum_{\gamma} \bar{R}_{\gamma\beta\alpha} h_{ip}^{\gamma} + \sum_{\eta} \bar{R}_{i\eta\alpha} h_{qp}^{\eta} - \sum_{\gamma} \bar{R}_{i\beta\gamma\alpha} h_{jp}^{\gamma} + \sum_{\eta} \bar{R}_{i\beta\eta\alpha} h_{qp}^{\eta}, \quad (9-1)$$

**证明:** 由定理 3.4, 3.5 立刻可得。

**引理 9.2:** 设  $x: M \rightarrow N^{n+p}$  是子流形,  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 则

$$\begin{aligned} \frac{\partial h_{ij}^{\alpha}}{\partial t} &= V_{,ij}^{\alpha} + \sum_p h_{ij,p}^{\alpha} V^p + \sum_p h_{pj}^{\alpha} L_i^p + \sum_p h_{ip}^{\alpha} L_j^p - \sum_{\beta} h_{ij}^{\beta} L_{\beta}^{\alpha} \\ &\quad + \sum_{\beta} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} - \sum_{\beta} \bar{R}_{i\beta j}^{\alpha} V^{\beta}, \end{aligned} \quad (9-2)$$

$$\frac{\partial}{\partial t} H^{\alpha} = \frac{1}{n} \Delta V^{\alpha} + \sum_i H_{,i}^{\alpha} V^i - H^{\beta} L_{\beta}^{\alpha} + \frac{1}{n} S_{\alpha\beta} V^{\beta} + \frac{1}{n} \bar{R}_{\alpha\beta}^{\gamma} V^{\beta}, \quad (9-3)$$

$$\frac{\partial S}{\partial t} = \sum 2h_{kl}^{\beta} V_{,kl}^{\beta} + \sum_i S_{,i} V^i + \sum 2S_{\gamma\gamma\beta} V^{\beta} - \sum 2h_{kl}^{\gamma} \bar{R}_{kl\beta}^{\gamma} V^{\beta}, \quad (9-4)$$

$$\begin{aligned} \frac{\partial S_{\alpha\beta\beta}}{\partial t} &= V_{,ij}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} V_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} V_{,ki}^{\beta} \\ &\quad + S_{\alpha\beta\beta, i} V^i + S_{\gamma\beta\beta} L_{\alpha}^{\gamma} + S_{\alpha\gamma\beta\beta} V^{\gamma} + S_{\alpha\beta\gamma\beta} V^{\gamma} + S_{\alpha\beta\beta\gamma} V^{\gamma} \\ &\quad - (\bar{R}_{ij\gamma}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} \bar{R}_{jk\gamma}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} \bar{R}_{ki\gamma}^{\beta}) V^{\gamma}, \end{aligned} \quad (9-5)$$

$$\begin{aligned} \frac{\partial \bar{R}_{i\beta j\alpha}}{\partial t} &= \sum_{\gamma} \bar{R}_{i\beta\alpha; \gamma} V^{\gamma} + \sum_p \bar{R}_{i\beta\alpha; p} V^p \\ &\quad + \sum_q \bar{R}_{q\beta\alpha} L_i^q + \sum_{\gamma} \bar{R}_{\gamma\beta\alpha} (V_{,i}^{\gamma} + h_{ip}^{\gamma} V^p) \\ &\quad - \sum_q \bar{R}_{i\eta\alpha} (V_{,q}^{\beta} + h_{qp}^{\beta} V^p) + \sum_{\gamma} \bar{R}_{i\gamma\alpha} L_{\beta}^{\gamma} \end{aligned}$$

$$\begin{aligned}
& + \sum_q \bar{R}_{i\beta q\alpha} L_j^q + \sum_\gamma \bar{R}_{i\beta\gamma\alpha} (V_j^\gamma + h_{j\beta}^\gamma V^\beta) \\
& - \sum_q \bar{R}_{i\beta j\alpha} (V_{,q}^\alpha + h_{q\beta}^\alpha V^\beta) + \sum_\gamma \bar{R}_{i\beta j\gamma} L_\alpha^\gamma
\end{aligned} \quad (9-6)$$

$$\begin{aligned}
& = \sum_\gamma \bar{R}_{i\beta j\alpha; \gamma} V^\gamma + \sum_p (\bar{R}_{i\beta j\alpha, p} - \sum_\gamma \bar{R}_{\gamma\beta j\alpha} h_{ip}^\gamma + \sum_q \bar{R}_{i\beta j\alpha} h_{qp}^\beta \\
& - \sum_\gamma \bar{R}_{i\beta\gamma\alpha} h_{j\beta}^\gamma + \sum_q \bar{R}_{i\beta j\alpha} h_{qp}^\beta) V^p \\
& + \sum_q \bar{R}_{q\beta j\alpha} L_i^q + \sum_\gamma \bar{R}_{\gamma\beta j\alpha} (V_{,i}^\gamma + h_{i\beta}^\gamma V^\beta) \\
& - \sum_q \bar{R}_{i\beta j\alpha} (V_{,q}^\beta + h_{q\beta}^\beta V^\beta) + \sum_\gamma \bar{R}_{i\gamma j\alpha} L_\beta^\gamma \\
& + \sum_q \bar{R}_{i\beta q\alpha} L_j^q + \sum_\gamma \bar{R}_{i\beta\gamma\alpha} (V_j^\gamma + h_{j\beta}^\gamma V^\beta) \\
& - \sum_q \bar{R}_{i\beta j\alpha} (V_{,q}^\alpha + h_{q\beta}^\alpha V^\beta) + \sum_\gamma \bar{R}_{i\beta j\gamma} L_\alpha^\gamma.
\end{aligned} \quad (9-7)$$

证明: 由定理 3.3, 推论 3.4 立刻可得。

在上面的引理之中, 因为  $L_i^j$ ,  $L_\alpha^\beta$  不是张量, 在计算过程之中可以去掉但不会影响计算结果, 所以我们可以简化上面的引理为

**引理 9.3:** 设  $x: M \rightarrow N^{n+p}$  是子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 则

$$\frac{\partial h_{ij}^\alpha}{\partial t} = V_{,ij}^\alpha + \sum_p h_{ij, p}^\alpha V^p + \sum_{p\beta} h_{ip}^\alpha h_{j\beta}^\beta V^\beta - \sum_\beta \bar{R}_{i\beta j}^\alpha V^\beta, \quad (9-8)$$

$$\frac{\partial}{\partial t} H^\alpha = \frac{1}{n} \Delta V^\alpha + \sum_i H_{,i}^\alpha V^i + \frac{1}{n} S_{\alpha\beta} V^\beta + \frac{1}{n} \bar{R}_{\alpha\beta}^\gamma V^\beta, \quad (9-9)$$

$$\frac{\partial S}{\partial t} = \sum 2h_{kl}^\beta V_{,kl}^\beta + \sum_i S_{,i} V^i + \sum 2S_{\gamma\beta} V^\beta - \sum 2h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta, \quad (9-10)$$

$$\begin{aligned}
\frac{\partial S_{\alpha\beta\beta}}{\partial t} & = V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta \\
& + S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma \\
& - (\bar{R}_{i\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma,
\end{aligned} \quad (9-11)$$

$$\begin{aligned}
\frac{\partial \bar{R}_{i\beta j\alpha}}{\partial t} & = \sum_\gamma \bar{R}_{i\beta j\alpha; \gamma} V^\gamma + \sum_p \bar{R}_{i\beta j\alpha, p} V^p \\
& + \sum_\gamma \bar{R}_{\gamma\beta j\alpha} (V_{,i}^\gamma + h_{i\beta}^\gamma V^\beta) - \sum_q \bar{R}_{i\beta j\alpha} (V_{,q}^\beta + h_{q\beta}^\beta V^\beta) \\
& + \sum_\gamma \bar{R}_{i\beta\gamma\alpha} (V_j^\gamma + h_{j\beta}^\gamma V^\beta) - \sum_q \bar{R}_{i\beta j\alpha} (V_{,q}^\alpha + h_{q\beta}^\alpha V^\beta)
\end{aligned} \quad (9-12)$$

$$\begin{aligned}
& = \sum_\gamma \bar{R}_{i\beta j\alpha; \gamma} V^\gamma + \sum_p \bar{R}_{i\beta j\alpha, p} V^p \\
& + \sum_\gamma \bar{R}_{\gamma\beta j\alpha} V_{,i}^\gamma - \sum_q \bar{R}_{i\beta j\alpha} V_{,q}^\beta + \sum_\gamma \bar{R}_{i\beta\gamma\alpha} V_j^\gamma - \sum_q \bar{R}_{i\beta j\alpha} V_{,q}^\alpha.
\end{aligned} \quad (9-13)$$



由前面一节的第一变分公式与以上两个引理, 我们可以计算  $GD_{(n, F)}$  泛函和  $GD_{(n, F, \epsilon)}$  泛函的第二变分公式, 这是讨论  $GD_{(n, F)}$  子流形和  $GD_{(n, F, \epsilon)}$  子流形稳定性的基础。我们分别计算之。

首先对于  $GD_{(n, F)}$  泛函, 我们已经计算

$$\begin{aligned} \frac{\partial}{\partial t} GD_{(n, F)}(x_i) &= \int_{M_i} F'(S) \frac{\partial}{\partial t}(S) + F(S) (V_{,i}^i - nH^\alpha V^\alpha) dv \\ &= \int_{M_i} F'(S) (2h_{ij}^\alpha V_{,ij}^\alpha + S_{,i} V^i + 2S_{\alpha\beta\beta} V^\alpha - 2h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V^\alpha) \\ &\quad + F(S) (V_{,i}^i - nH^\alpha V^\alpha) dv \\ &= \int_M (2F'(S) h_{ij}^\alpha)_{,ij} V^\alpha + F'(S) S_{,i} V^i + 2F'(S) S_{\alpha\beta\beta} V^\alpha \\ &\quad - 2F'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V^\alpha - F'(S) S_{,i} V^i - nH^\alpha F(S) V^\alpha \\ &= \int_M [(2F'(S) h_{ij}^\alpha)_{,ij} + 2F'(S) S_{\alpha\beta\beta} - 2F'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta - nF(S) H^\alpha] V^\alpha dv. \end{aligned}$$

于是, 在  $x: M \rightarrow N$  为  $GD_{(n, F)}$  子流形的假设之下 (即是  $GD_{(n, F)}$  在 0 点的一阶变分为 0), 泛函的第二变分计算为

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \Big|_{t=0} GD_{(n, F)}(x_i) &= \int_M \left[ (2F''(S) h_{ij}^\alpha \frac{\partial S}{\partial t} + 2F'(S) \frac{\partial h_{ij}^\alpha}{\partial t})_{,ij} \right. \\ &\quad + 2F''(S) S_{\alpha\beta\beta} \frac{\partial S}{\partial t} + 2F'(S) \frac{\partial S_{\alpha\beta\beta}}{\partial t} \\ &\quad - 2F''(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta \frac{\partial S}{\partial t} - 2F'(S) \bar{R}_{ij\alpha}^\beta \frac{\partial h_{ij}^\beta}{\partial t} - 2F'(S) h_{ij}^\beta \frac{\partial \bar{R}_{ij\alpha}^\beta}{\partial t} \\ &\quad \left. - nF'(S) H^\alpha \frac{\partial S}{\partial t} - nF(S) \frac{\partial H^\alpha}{\partial t} \right] V^\alpha dv. \\ &= \int_M \left[ (2F''(S) h_{ij}^\alpha (2h_{kl}^\beta V_{,kl}^\beta + S_{,p} V^p + 2S_{\gamma\gamma\beta} V^\beta - 2h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \right. \\ &\quad + 2F'(S) (V_{,ij}^\alpha + h_{ij,p}^\alpha V^p + h_{ip}^\alpha h_{pj}^\beta V^\beta - \bar{R}_{ij\beta}^\alpha V^\beta))_{,ij} \\ &\quad + 2F''(S) S_{\alpha\beta\beta} (2h_{kl}^\gamma V_{,kl}^\gamma + S_{,p} V^p + 2S_{\delta\delta\gamma} V^\gamma - 2h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\ &\quad + 2F'(S) (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta \\ &\quad + S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma \\ &\quad - (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma) \\ &\quad - 2F''(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta (2h_{kl}^\gamma V_{,kl}^\gamma + S_{,p} V^p + 2S_{\delta\delta\gamma} V^\gamma - 2h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\ &\quad - 2F'(S) \bar{R}_{ij\alpha}^\beta (V_{,ij}^\beta + h_{ij,p}^\beta V^p + h_{ip}^\beta h_{pj}^\gamma V^\gamma - \bar{R}_{ij\gamma}^\beta V^\gamma) \\ &\quad \left. - 2F'(S) h_{ij}^\beta (\bar{R}_{ij\alpha}^\beta V^\alpha + \sum_p \bar{R}_{ij\alpha, p}^\beta V^p \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{\gamma} \bar{R}_{\gamma\beta\alpha} V_{,i}^{\gamma} - \sum_{\gamma} \bar{R}_{\gamma\beta\alpha} V_{,q}^{\beta} + \sum_{\gamma} \bar{R}_{i\beta\gamma\alpha} V_{,j}^{\gamma} - \sum_{\gamma} \bar{R}_{i\beta\gamma\alpha} V_{,q}^{\alpha} ) \\
& - nF'(S) H^{\alpha} (2h_{kl}^{\beta} V_{,kl}^{\beta} + S_{,p} V^p + 2S_{\gamma\beta} V^{\beta} - 2h_{kl}^{\gamma} \bar{R}_{kl\beta}^{\gamma} V^{\beta}) \\
& - nF(S) \left( \frac{1}{n} \Delta V^{\alpha} + H_{,p}^{\alpha} V^p + \frac{1}{n} S_{\alpha\beta} V^{\beta} + \frac{1}{n} \bar{R}_{\alpha\beta}^{\gamma} V^{\beta} \right) ] V^{\alpha} dv.
\end{aligned}$$

通过分部积分公式, 我们得到

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, \mathbb{R})}(x_t) \\
& = \int_M 2F'(S) V_{,ij}^{\alpha} h_{ij}^{\alpha} (2h_{kl}^{\beta} V_{,kl}^{\beta} + S_{,p} V^p + 2S_{\gamma\beta} V^{\beta} - 2h_{kl}^{\gamma} \bar{R}_{kl\beta}^{\gamma} V^{\beta}) \\
& \quad + 2F'(S) V_{,ij}^{\alpha} (V_{,ij}^{\alpha} + h_{ij,p}^{\alpha} V^p + h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} - \bar{R}_{ij\beta}^{\alpha} V^{\beta}) \\
& \quad + 2F'(S) V^{\alpha} S_{\alpha\beta\beta} (2h_{kl}^{\gamma} V_{,kl}^{\gamma} + S_{,p} V^p + 2S_{\delta\gamma} V^{\gamma} - 2h_{kl}^{\delta} \bar{R}_{kl\gamma}^{\delta} V^{\gamma}) \\
& \quad + 2F'(S) V^{\alpha} (V_{,ij}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} V_{,jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} V_{,ki}^{\beta} \\
& \quad + S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^{\gamma} + S_{\alpha\beta\gamma\beta} V^{\gamma} + S_{\alpha\beta\beta\gamma} V^{\gamma} \\
& \quad - (\bar{R}_{ij\gamma}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} \bar{R}_{jk\gamma}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} \bar{R}_{ki\gamma}^{\beta}) V^{\gamma}) \\
& \quad - 2F'(S) V^{\alpha} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} (2h_{kl}^{\gamma} V_{,kl}^{\gamma} + S_{,p} V^p + 2S_{\delta\gamma} V^{\gamma} - 2h_{kl}^{\delta} \bar{R}_{kl\gamma}^{\delta} V^{\gamma}) \\
& \quad - 2F'(S) V^{\alpha} \bar{R}_{ij\alpha}^{\beta} (V_{,ij}^{\beta} + h_{ij,p}^{\beta} V^p + h_{ip}^{\beta} h_{pj}^{\gamma} V^{\gamma} - \bar{R}_{ij\gamma}^{\beta} V^{\gamma}) \\
& \quad - 2F'(S) V^{\alpha} h_{ij}^{\beta} (\bar{R}_{i\beta\alpha;\gamma} V^{\gamma} + \bar{R}_{i\beta\alpha,p} V^p \\
& \quad + \bar{R}_{\gamma\beta\alpha} V_{,i}^{\gamma} - \bar{R}_{\gamma\beta\alpha} V_{,q}^{\beta} + \bar{R}_{i\beta\gamma\alpha} V_{,j}^{\gamma} - \bar{R}_{i\beta\gamma\alpha} V_{,q}^{\alpha}) \\
& \quad - nF'(S) V^{\alpha} H^{\alpha} (2h_{kl}^{\beta} V_{,kl}^{\beta} + S_{,p} V^p + 2S_{\gamma\beta} V^{\beta} - 2h_{kl}^{\gamma} \bar{R}_{kl\beta}^{\gamma} V^{\beta}) \\
& \quad - nF(S) V^{\alpha} \left( \frac{1}{n} \Delta V^{\alpha} + H_{,p}^{\alpha} V^p + \frac{1}{n} S_{\alpha\beta} V^{\beta} + \frac{1}{n} \bar{R}_{\alpha\beta}^{\gamma} V^{\beta} \right) dv.
\end{aligned}$$

进一步整理得到

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, \mathbb{R})}(x_t) \\
& = \int_M F'(S) (4h_{kl}^{\beta} V_{,kl}^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 2S_{,p} V^p V_{,ij}^{\alpha} h_{ij}^{\alpha} + 4S_{\gamma\beta} V^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} - 4V_{,ij}^{\alpha} h_{ij}^{\alpha} h_{kl}^{\gamma} \bar{R}_{kl\beta}^{\gamma} V^{\beta}) \\
& \quad + F'(S) (2V_{,ij}^{\alpha} V_{,ij}^{\alpha} + 2V_{,ij}^{\alpha} h_{ij,p}^{\alpha} V^p + 2V_{,ij}^{\alpha} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} - 2V_{,ij}^{\alpha} \bar{R}_{ij\beta}^{\alpha} V^{\beta}) \\
& \quad + F'(S) (4V^{\alpha} S_{\alpha\beta\beta} h_{kl}^{\gamma} V_{,kl}^{\gamma} + 2V^{\alpha} S_{\alpha\beta\beta} S_{,p} V^p + 4V^{\alpha} S_{\alpha\beta\beta} S_{\delta\gamma} V^{\gamma} - 4V^{\alpha} S_{\alpha\beta\beta} h_{kl}^{\delta} \bar{R}_{kl\gamma}^{\delta} V^{\gamma}) \\
& \quad + F'(S) (2V^{\alpha} (V_{,ij}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} V_{,jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} V_{,ki}^{\beta}) \\
& \quad + 2V^{\alpha} (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^{\gamma} + S_{\alpha\beta\gamma\beta} V^{\gamma} + S_{\alpha\beta\beta\gamma} V^{\gamma}) \\
& \quad - 2V^{\alpha} (\bar{R}_{ij\gamma}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} \bar{R}_{jk\gamma}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} \bar{R}_{ki\gamma}^{\beta}) V^{\gamma}) \\
& \quad - F'(S) (4V^{\alpha} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} h_{kl}^{\gamma} V_{,kl}^{\gamma} + 2V^{\alpha} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} S_{,p} V^p + 4V^{\alpha} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} S_{\delta\gamma} V^{\gamma} - 4V^{\alpha} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} h_{kl}^{\delta} \bar{R}_{kl\gamma}^{\delta} V^{\gamma}))
\end{aligned}$$

$$\begin{aligned}
& -F'(S)(2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& -F'(S)(2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha;\gamma} V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha,p} V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{\gamma\beta\alpha} V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha}^\gamma V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma\alpha} V_j^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma q} V_{,q}^\alpha) \\
& -F'(S)(2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta - 2nV^\alpha H^\alpha \bar{R}_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& -F(S)(V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dv.
\end{aligned}$$

因此我们得到了关于  $GD_{(n,F)}$  子流形和  $GD_{(n,F,\epsilon)}$  的二阶变分的如下定理。

**定理 9.1:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n,F)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n,F)}(x_i) \\
& = \int_M F'(S)(4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + F'(S)(2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
& + F'(S)(4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& + F'(S)(2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma) \\
& - F'(S)(4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma \\
& - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - F'(S)(2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - F'(S)(2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha;\gamma} V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha,p} V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{\gamma\beta\alpha} V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha}^\gamma V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma\alpha} V_j^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma q} V_{,q}^\alpha) \\
& - F'(S)(2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta - 2nV^\alpha H^\alpha \bar{R}_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& - F(S)(V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dv.
\end{aligned}$$

**定理 9.2:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n,F,\epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n,F,\epsilon)}(x_i) \\
& = \int_M F'(S + \epsilon)(4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + F'(S + \epsilon)(2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta)
\end{aligned}$$

$$\begin{aligned}
& + F'(S + \epsilon) (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& + F'(S + \epsilon) [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
& - F'(S + \epsilon) (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma \\
& - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - F'(S + \epsilon) (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij, p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - F'(S + \epsilon) (2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha\gamma}^\beta V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha, p}^\beta V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\gamma\alpha}^\beta V_{,j}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ijq}^\beta V_{,q}^\alpha) \\
& - F'(S + \epsilon) (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& - F(S + \epsilon) (V^\alpha \Delta V^\alpha + nV^\alpha H^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dv.
\end{aligned}$$

**定理 9.3:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n, F)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
& = \int_M F'(S) (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + F'(S) (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij, p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
& + F'(S) (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + F'(S) [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3, i} V^i + 3P_4 f) \\
& - 2f^2 (\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
& - F'(S) (4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 \\
& - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) - F'(S) (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij, p} V^p \\
& + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
& - F'(S) (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1); (n+1)} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1), p} V^p + 2f h_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
& - 2f h_{ij} \bar{R}_{i(n+1)j(n+1), q} + 2f h_{ij}^\beta \bar{R}_{i(n+1)(n+1)(n+1)j} f_{,j} - 2f h_{ij} \bar{R}_{i(n+1)j(n+1)q} f_{,q}) \\
& - F'(S) (2nf H h_{kl} f_{,kl} + nf H S_{,p} V^p + 2nf^2 H P_3 - 2nf^2 H h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& - F(S) (f \Delta f + nf H_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)ii(n+1)}) dv.
\end{aligned}$$

**定理 9.4:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n, F, \epsilon)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\
&= \int_M F''(S + \epsilon) (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad + F'(S + \epsilon) (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij, p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
&\quad + F''(S + \epsilon) (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad + F'(S + \epsilon) [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3,i} V^i + 3P_4 f) \\
&\quad - 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
&\quad - F''(S + \epsilon) (4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 \\
&\quad - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad - F'(S + \epsilon) (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij, p} V^p \\
&\quad + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
&\quad - F''(S + \epsilon) (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1);(n+1)} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1), p} V^p + 2f h_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
&\quad - 2f h_{ij} \bar{R}_{ij(n+1)} f_{,q} + 2f h_{ij} \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2f h_{ij} \bar{R}_{i(n+1)jq} f_{,q}) \\
&\quad - F'(S + \epsilon) (2nf H h_{kl} f_{,kl} + nf H S_{,p} V^p + 2nf^2 H P_3 - 2nf^2 H h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad - F(S + \epsilon) (f \Delta f + nf H_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)ii(n+1)}) dv.
\end{aligned}$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\begin{aligned}
\bar{R}_{ABCD} &= -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^{\beta} = -c\delta_{ij}\delta_{\alpha\beta}, \\
\bar{R}_{AB}^{\gamma} &= \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c\sum_i \delta_{Ai}\delta_{iB}, \\
\bar{R}_{AB}^{\perp} &= \sum_{\alpha} \bar{R}_{A\alpha\alpha B} = \sum_{\alpha} -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c\sum_{\alpha} \delta_{A\alpha}\delta_{B\alpha}, \\
\bar{R}_{\alpha\beta}^{\gamma} &= nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.
\end{aligned}$$

**定理 9.5:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n, F)}$  子流形,  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
&= \int_M F''(S) (4h_{kl}^{\beta} V_{,kl}^{\alpha} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 2S_{,p} V^p V_{,ij}^{\alpha} h_{ij}^{\alpha} + 4S_{\gamma\gamma\beta} V^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 4ncH^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} V^{\beta}) \\
&\quad + F'(S) (2V_{,ij}^{\alpha} V_{,ij}^{\alpha} + 2V_{,ij}^{\alpha} h_{ij, p}^{\alpha} V^p + 2V_{,ij}^{\alpha} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} + 2c\Delta(V^{\alpha}) V^{\alpha}) \\
&\quad + F''(S) (4V^{\alpha} S_{\alpha\beta\beta} h_{kl}^{\gamma} V_{,kl}^{\gamma} + 2V^{\alpha} S_{\alpha\beta\beta} S_{,p} V^p + 4V^{\alpha} S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^{\gamma} + 4ncH^{\gamma} V^{\alpha} S_{\alpha\beta\beta} V^{\gamma}) \\
&\quad + F'(S) [2V^{\alpha} (V_{,ij}^{\alpha} h_{jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} V_{,jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} V_{,ki}^{\beta})
\end{aligned}$$

$$\begin{aligned}
& + 2V^\alpha (S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta}) ] \\
& - F'(S) (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\delta\gamma} V^\gamma) \\
& - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\
& - F'(S) (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha) \\
& - F'(S) (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\gamma\beta} V^\beta + 2n^2 cV^\alpha H^\alpha H^\beta V^\beta) \\
& - F(S) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + ncV^\alpha V^\alpha) dv.
\end{aligned}$$

**定理 9.6:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n, F, \epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\
& = \int_M F'(S + \epsilon) (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 4ncH^\beta V_{,ij}^\alpha h_{ij}^\alpha V^\beta) \\
& + F'(S + \epsilon) [(2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha) V^\alpha] \\
& + F'(S + \epsilon) (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta\beta} V^\gamma) \\
& + F'(S + \epsilon) [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta}) ] \\
& - F'(S + \epsilon) (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\delta\gamma} V^\gamma \\
& - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\
& - F'(S + \epsilon) (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha) \\
& - F'(S + \epsilon) (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\gamma\beta} V^\beta + 2n^2 cV^\alpha H^\alpha H^\beta V^\beta) \\
& - F(S + \epsilon) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + ncV^\alpha V^\alpha) dv.
\end{aligned}$$

**定理 9.7:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n, F)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
& = \int_M F'(S) (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} + 4nc f f_{,ij} h_{ij} H) \\
& + F'(S) (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij} h_{ij, p} V^p + 2f f_{,ij} h_{ip} h_{pj} + 2cf \Delta f) \\
& + F'(S) (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 + 4nc f^2 P_3 H) \\
& + F'(S) [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3, i} V^i + 3P_4 H) \\
& - 2f^2 (-c\delta_{ij} h_{jk} h_{ki} - c\delta_{jk} h_{ij} h_{ki} - c\delta_{ik} h_{ij} h_{jk})] \\
& - F'(S) (-4ncH f h_{kl} f_{,kl} - 2ncf H S_{,p} V^p - 4ncf^2 H P_3 - 4n^2 c^2 f^2 H^2)
\end{aligned}$$

$$\begin{aligned}
& -F'(S)(-2cf\Delta f - 2ncfH_{,p}V^p - 2cf^2S - 2c^2nf^2) \\
& -F'(S)(2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 + 2n^2cf^2H^2) \\
& -F(S)(f\Delta f + nfH_{,p}V^p + f^2P_2 + nc f^2) dv.
\end{aligned}$$

**定理 9.8:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n, F, \epsilon)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_i) \\
& = \int_M F'(S + \epsilon) (4h_{kl}f_{,kl}f_{,ij}h_{ij} + 2S_{,p}V^p f_{,ij}h_{ij} + 4P_3 f f_{,ij}h_{ij} + 4nc f f_{,ij}h_{ij}H) \\
& + F'(S + \epsilon) (2f_{,ij}f_{,ij} + 2f_{,ij}h_{ij,p}V^p + 2ff_{,ij}h_{ip}h_{pj} + 2cf\Delta f) \\
& + F'(S + \epsilon) (4fP_3 h_{kl}f_{,kl} + 2fP_3 S_{,p}V^p + 4f^2P_3P_3 + 4ncf^2P_3H) \\
& + F'(S + \epsilon) [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4f) \\
& - 2f^2(-c\delta_{ij}h_{jk}h_{ki} - c\delta_{jk}h_{ij}h_{ki} - c\delta_{ik}h_{ij}h_{jk})] \\
& - F'(S + \epsilon) (-4ncHf h_{kl}f_{,kl} - 2ncfHS_{,p}V^p - 4ncf^2HP_3 - 4n^2c^2f^2H^2) \\
& - F'(S + \epsilon) (-2cf\Delta f - 2ncfH_{,p}V^p - 2cf^2S - 2c^2nf^2) \\
& - F'(S + \epsilon) (2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 + 2n^2cf^2H^2) \\
& - F(S + \epsilon) (f\Delta f + nfH_{,p}V^p + f^2P_2 + nc f^2) dv.
\end{aligned}$$

**注释 9.1:** 特别注意上面诸定理之中黎曼张量  $\bar{R}_{\alpha\beta\gamma\delta}$  分别在流形  $N$  和流形  $M$  上的拉回丛  $x^*TN$  上的两个协变导数  $\bar{R}_{\alpha\beta\gamma\delta;p}$  和  $\bar{R}_{\alpha\beta\gamma\delta}$  的区别。

## 9.2 $GD_{(n, r)}$ 泛函的第二变分公式

我们知道当  $F(u) = u^r$  时, 泛函  $GD_{n, F}$  和  $GD_{(n, F, \epsilon)}$  分别变为

$$GD_{(n, r)} = \int_M S^r dv, \quad GD_{(n, r, \epsilon)} = \int_M (S + \epsilon)^r dv$$

针对此类重要的特殊情形的计算, 可以直接利用第一节的定理得到, 于是我们有

**定理 9.9:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n, r)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_i) \\
& = \int_M r(r-1)S^{r-2} (4h_{kl}^\beta V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p}V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta}V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + rS^{r-1} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
& + r(r-1)S^{r-2} (4V_{\alpha\beta\gamma}^\alpha S_{\alpha\beta\gamma} h_{kl}^\gamma h_{kl}^\gamma + 2V_{\alpha\beta\gamma}^\alpha S_{\alpha\beta\gamma} S_{,p}V^p + 4V_{\alpha\beta\gamma}^\alpha S_{\alpha\beta\gamma} S_{\delta\delta\gamma} V^\gamma - 4V_{\alpha\beta\gamma}^\alpha S_{\alpha\beta\gamma} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma)
\end{aligned}$$

$$\begin{aligned}
& + rS^{r-1} [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
& - r(r-1)S^{r-2} (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma \\
& - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - rS^{r-1} (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - rS^{r-1} (2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha;\gamma} V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha,p} V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{\gamma\beta\alpha} V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha} V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma\alpha} V_j^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma q} V_q^\alpha) \\
& - rS^{r-1} (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\gamma\beta} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& - S' (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dv.
\end{aligned}$$

**定理 9.10:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n,r,\epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD(n, F, \epsilon)(x_t) \\
& = \int_M r(r-1)(S+\epsilon)^{r-2} (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + r(S+\epsilon)^{r-1} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
& + r(r-1)(S+\epsilon)^{r-2} (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& + Fr(S+\epsilon)^{r-1} [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& + 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
& - r(r-1)(S+\epsilon)^{r-2} (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma \\
& - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - r(S+\epsilon)^{r-1} (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - r(S+\epsilon)^{r-1} (2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha;\gamma} V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha,p} V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{\gamma\beta\alpha} V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha} V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma\alpha} V_j^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma q} V_q^\alpha) \\
& - r(S+\epsilon)^{r-1} (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\gamma\beta} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& - (S+\epsilon)' (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dV.
\end{aligned}$$

**定理 9.11:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n,r)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为



$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_i) \\
&= \int_M r(r-1) S^{r-2} (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad + r S^{r-1} (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij, p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
&\quad + r(r-1) S^{r-2} (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad + r S^{r-1} [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3, i} V^i + 3P_4 f) \\
&\quad - 2f^2 (\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
&\quad - r(r-1) S^{r-2} (4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 \\
&\quad - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad - r S^{r-1} (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij, p} V^p \\
&\quad + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
&\quad - r S^{r-1} (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1), (n+1)} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1), p} V^p + 2f h_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
&\quad - 2f h_{ij} \bar{R}_{ij(n+1)} f_{,q} + 2f h_{ij} \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} f_{,q}) \\
&\quad - r S^{r-1} (2nf H h_{kl} f_{,kl} + n f H S_{,p} V^p + 2n f^2 H P_3 - 2n f^2 H h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad - S^r (f \Delta f + n f H_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)ii(n+1)}) dv.
\end{aligned}$$

**定理 9.12:** 假设  $x: M^n \rightarrow M^{n+1}$  是一般原流形之中的  $GD_{(n, r, \epsilon)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_i) \\
&= \int_M r(r-1) (S + \epsilon)^{r-2} (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} \\
&\quad + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad + r (S + \epsilon)^{r-1} (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij, p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
&\quad + r(r-1) (S + \epsilon)^{r-2} (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad + r (S + \epsilon)^{r-1} [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3, i} V^i + 3P_4 f) \\
&\quad - 2f^2 (\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
&\quad - r(r-1) (S + \epsilon)^{r-2} (4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p \\
&\quad + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
&\quad - r (S + \epsilon)^{r-1} (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij, p} V^p \\
&\quad + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)})
\end{aligned}$$

$$\begin{aligned}
& -r(S+\epsilon)^{r-1}(2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1);(n+1)} + 2fh_{ij} \bar{R}_{i(n+1)j(n+1),p} V^p + 2fh_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
& - 2fh_{ij} \bar{R}_{i(n+1)j(n+1),q} f_{,q} + 2fh_{ij} \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2fh_{ij} \bar{R}_{i(n+1)j(n+1),q} f_{,q}) \\
& -r(S+\epsilon)^{r-1}(2nfHh_{kl} f_{,kl} + nfHS_{,p} V^p + 2nf^2 HP_3 - 2nf^2 Hh_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& - (S+\epsilon)^r (f\Delta f + nfH_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)i(n+1)}) dv.
\end{aligned}$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\begin{aligned}
\bar{R}_{ABCD} &= -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^{\beta} = -c\delta_{ij}\delta_{\alpha\beta}, \\
\bar{R}_{AB}^{\gamma} &= \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c\sum_i \delta_{Ai}\delta_{iB}, \\
\bar{R}_{AB}^{\perp} &= \sum_{\alpha} \bar{R}_{A\alpha\alpha B} = \sum_{\alpha} -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c\sum_{\alpha} \delta_{A\alpha}\delta_{B\alpha}, \\
\bar{R}_{\alpha\beta}^{\gamma} &= nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.
\end{aligned}$$

**定理 9.13:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n,r)}$  子流形,  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{n,F}(x_t) \\
&= \int_M r(r-1)S^{r-2} (4h_{kl}^{\beta} V_{,kl}^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 2S_{,p} V^p V_{,ij}^{\alpha} h_{ij}^{\alpha} + 4S_{\gamma\gamma\beta} V^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 4ncH^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} V^{\beta}) \\
&+ rS^{r-1} (2V_{,ij}^{\alpha} + 2V_{,ij}^{\alpha} h_{ij,p}^{\alpha} V^p + 2V_{,ij}^{\alpha} h_{ip}^{\alpha} h_{pj}^{\beta} V^{\beta} + 2c\Delta(V^{\alpha}) V^{\alpha}) \\
&+ r(r-1)S^{r-2} (4V^{\alpha} S_{\alpha\beta\beta} h_{kl}^{\gamma} V_{,kl}^{\gamma} + 2V^{\alpha} S_{\alpha\beta\beta} S_{,p} V^p + 4V^{\alpha} S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^{\gamma} + 4ncH^{\gamma} V^{\alpha} S_{\alpha\beta\beta} V^{\gamma}) \\
&+ rS^{r-1} [2V^{\alpha} (V_{,ij}^{\beta} h_{jk}^{\beta} h_{jk}^{\beta} + h_{ij}^{\alpha} V_{,jk}^{\beta} h_{ki}^{\beta} + h_{ij}^{\alpha} h_{jk}^{\beta} V_{,ki}^{\beta}) \\
&+ 2V^{\alpha} (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^{\gamma} + S_{\alpha\beta\gamma\beta} V^{\gamma} + S_{\alpha\beta\beta\gamma} V^{\gamma}) \\
&- 2V^{\alpha} (-cV^{\alpha} S - 2cV^{\beta} S_{\alpha\beta})] \\
&- r(r-1)S^{r-2} (-4ncV^{\alpha} H^{\alpha} h_{kl}^{\gamma} V_{,kl}^{\gamma} - 2ncV^{\alpha} H^{\alpha} S_{,p} V^p - 4ncV^{\alpha} H^{\alpha} S_{\delta\delta\gamma} V^{\gamma} \\
&- 4n^2 c^2 V^{\alpha} H^{\alpha} H^{\gamma} V^{\gamma}) \\
&- rS^{r-1} (-2cV^{\alpha} \Delta(V^{\alpha}) - 2ncV^{\alpha} H^{\alpha}_{,p} V^p - 2cV^{\alpha} S_{\alpha\gamma} V^{\gamma} - 2nc^2 V^{\alpha} V^{\alpha}) \\
&- rS^{r-1} (2nV^{\alpha} H^{\alpha} h_{kl}^{\beta} V_{,kl}^{\beta} + nV^{\alpha} H^{\alpha} S_{,p} V^p + 2nV^{\alpha} H^{\alpha} S_{\gamma\gamma\beta} V^{\beta} + 2n^2 cV^{\alpha} H^{\alpha} H^{\beta} V^{\beta}) \\
&- S^r (V^{\alpha} \Delta V^{\alpha} + nV^{\alpha} H^{\alpha}_{,p} V^p + V^{\alpha} S_{\alpha\beta} V^{\beta} + ncV^{\alpha} V^{\alpha}) dv.
\end{aligned}$$

**定理 9.14:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n,r,\epsilon)}$  子流形,  $V = V^i e_i + V^{\alpha} e_{\alpha}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n,F,\epsilon)}(x_t) \\
&= \int_M r(r-1)(S+\epsilon)^{r-2} (4h_{kl}^{\beta} V_{,kl}^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 2S_{,p} V^p V_{,ij}^{\alpha} h_{ij}^{\alpha} + 2S_{,p} V^p V_{,ij}^{\alpha} h_{ij}^{\alpha} \\
&+ 4S_{\gamma\gamma\beta} V^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} + 4ncH^{\beta} V_{,ij}^{\alpha} h_{ij}^{\alpha} V^{\beta})
\end{aligned}$$

$$\begin{aligned}
& + r(S + \epsilon)^{r-1} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha)V^\alpha) \\
& + r(r-1)(S + \epsilon)^{r-2} (4V_{\alpha\beta}^\alpha S_{\alpha\beta}^\gamma h_{kl}^\gamma V_{,kl}^\gamma + 2V_{\alpha\beta}^\alpha S_{\alpha\beta}^\gamma S_{,p} V^p \\
& + 4V_{\alpha\beta}^\alpha S_{\alpha\beta}^\gamma S_{\delta\delta}^\gamma V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta}^\gamma V^\gamma) \\
& + r(S + \epsilon)^{r-1} [2V_{,ij}^\alpha (V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,k}^\beta i) \\
& + 2V_{\alpha\beta}^\alpha S_{\alpha\beta}^\gamma V^i + S_{\alpha\gamma\beta\beta}^\gamma V^\gamma + S_{\alpha\beta\gamma\beta}^\gamma V^\gamma + S_{\alpha\beta\beta\gamma}^\gamma V^\gamma) \\
& - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta})] \\
& - r(r-1)(S + \epsilon)^{r-2} (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\delta}^\gamma V^\gamma \\
& - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\
& - r(S + \epsilon)^{r-1} (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma}^\gamma V^\gamma - 2nc^2 V^\alpha V^\alpha) \\
& - r(S + \epsilon)^{(r-1)} (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p \\
& + 2nV^\alpha H^\alpha S_{\gamma\gamma}^\beta V^\beta + 2n^2 c V^\alpha H^\alpha H^\beta + V^\beta) \\
& - (S + \epsilon)^r (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta}^\beta V^\beta + ncV^\alpha V^\alpha) dv.
\end{aligned}$$

**定理 9.15:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n,r)}$  超曲面,  $V = V e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n,r)}(x_t) \\
& = \int_M r(r-1) S^{r-2} (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} + 4nc f f_{,ij} h_{ij} H) \\
& + rS^{r-1} (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij,p} V^p + 2f f_{,ij} h_{ip} h_{pj} + 2cf \Delta f) \\
& + r(r-1) S^{r-2} (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 + 4nc f^2 P_3 H) \\
& + rS^{r-1} [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3,i} V^i + 3P_4 f) \\
& - 2f^2 (-c\delta_{ij} h_{jk} h_{ki} - c\delta_{jk} h_{ij} h_{ki} - c\delta_{ik} h_{ij} h_{jk})] \\
& - r(r-1) S^{r-2} (-4nc f H h_{kl} f_{,kl} - 2nc f H S_{,p} V^p - 4nc f^2 H P_3 - 4n^2 c^2 f^2 H^2) \\
& - rS^{r-1} (-2cf \Delta f - 2nc f H_{,p} V^p - 2cf^2 S - 2c^2 n f^2) \\
& - rS^{r-1} (2nf H h_{kl} f_{,kl} + nf H S_{,p} V^p + 2nf^2 H P_3 + 2n^2 c f^2 H^2) \\
& - S^r (f \Delta f + nf H_{,p} V^p + f^2 P_2 + nc f^2) dv.
\end{aligned}$$

**定理 9.16:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n,r,\epsilon)}$  超曲面,  $V = V e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n,r,\epsilon)}(x_t) \\
& = \int_M r(r-1)(S + \epsilon)^{r-2} (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} + 4nc f f_{,ij} h_{ij} H) \\
& + r(S + \epsilon)^{r-1} (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij,p} V^p + 2f f_{,ij} h_{ip} h_{pj} + 2cf \Delta f) \\
& + Fr(r-1)(S + \epsilon)^{r-2} (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 + 4nc f^2 P_3 H)
\end{aligned}$$

$$\begin{aligned}
& + r(S + \epsilon)^{r-1} [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4f) \\
& - 2f^2(-c\delta_{ij}h_{jk}h_{ki} - c\delta_{jk}h_{ij}h_{ki} - c\delta_{ik}h_{ij}h_{jk})] \\
& - r(r-1)(S + \epsilon)^{r-2}(-4ncHf h_{kl}f_{,kl} - 2ncfHS_{,p}V^p - 4ncf^2HP_3 - 4n^2c^2f^2H^2) \\
& - r(S + \epsilon)^{r-1}(-2cf\Delta f - 2ncfH_{,p}V^p - 2cf^2S - 2c^2nf^2) \\
& - r(S + \epsilon)^{r-1}(2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 + 2n^2cf^2H^2) \\
& - (S + \epsilon)^r(f\Delta f + nfH_{,p}V^p + f^2P_2 + nc f^2) \, dv.
\end{aligned}$$

**注释 9.2:** 特别注意上面诸定理之中黎曼张量  $\bar{R}_{\alpha\beta\gamma\delta}$  分别在流形  $N$  和流形  $M$  上的拉回丛  $x^*TN$  上的两个协变导数  $\bar{R}_{\alpha\beta\gamma\delta;p}$  和  $\bar{R}_{\alpha\beta\gamma\delta,p}$  的区别。

### 9.3 $GD_{(n,E)}$ 泛函的第二变分公式

我们知道当  $F(u) = e^u$  时, 泛函  $GD_{(n,F)}$  和  $GD_{(n,F,\epsilon)}$  分别变为

$$GD_{(n,E)} = \int_M e^S dv, \quad GD_{(n,E,\epsilon)} = \int_M e^{S+\epsilon} dv$$

显然  $GD_{(n,E)}$  和  $GD_{(n,E,\epsilon)}$  在相差一个微小摄动的意义下是相同的, 因此针对此类重要的特殊情形的计算, 可以直接利用第一节的定理得到, 于是我们有

**定理 9.17:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n,E)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n,F)}(x_t) \\
& = \int_M e^S (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta} V_{,ij}^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + e^S (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
& + e^S (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& + e^S [2V^\alpha (V_{,ij}^\beta h_{jk}^\beta h_{kl}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{kl}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
& - e^S (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma \\
& - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - e^S (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - e^S (2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha;\gamma} V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha,p} V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{\gamma\beta\alpha} V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha} V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma\alpha} V_{,j}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma q} V_{,q}^\alpha)
\end{aligned}$$

$$\begin{aligned}
& -e^S(2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& -e^S(V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\tau V^\beta) dv.
\end{aligned}$$

**定理 9.18:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n,E)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \frac{\partial^2}{\partial t^2} \Big|_{t=0} GD_{(n,F)}(x_t) \\
& = \int_M e^S (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + e^S (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij,p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
& + e^S (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + e^S [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3,i} V^i + 3P_4 f) \\
& - 2f^2 (\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
& - e^S (4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 \\
& - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& - e^S (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij,p} V^p \\
& + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
& - e^S (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} f_{,i(n+1)} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1),p} V^p + 2f h_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
& - 2f h_{ij} \bar{R}_{ij(n+1)} f_{,q} + 2f h_{ij}^\beta \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2f h_{ij} \bar{R}_{i(n+1)jq} f_{,q}) \\
& - e^S (2nf H h_{kl} f_{,kl} + n f H S_{,p} V^p + 2n f^2 H P_3 - 2n f^2 H h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& - e^S (f \Delta f + n f H_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)ii(n+1)}) dv.
\end{aligned}$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\begin{aligned}
\bar{R}_{ABCD} &= -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^\beta = -c\delta_{ij}\delta_{\alpha\beta}, \\
\bar{R}_{AB}^\tau &= \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c \sum_i \delta_{Ai}\delta_{iB}, \\
\bar{R}_{AB}^\perp &= \sum_\alpha \bar{R}_{A\alpha\alpha B} = \sum_\alpha -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c \sum_\alpha \delta_{A\alpha}\delta_{B\alpha}, \\
\bar{R}_{\alpha\beta}^\tau &= nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^\perp = pc\delta_{ij}.
\end{aligned}$$

**定理 9.19:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n,E)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
&= \int_M e^S (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 4ncH^\beta V_{,ij}^\alpha h_{ij}^\alpha V^\beta) \\
&\quad + e^S (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha) V^\alpha) \\
&\quad + e^S (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta\beta} V^\gamma) \\
&\quad + e^S [2V^\alpha (v_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{ij,ki}^\beta V^\beta) \\
&\quad + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta})] \\
&\quad - e^S (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\delta\gamma} V^\gamma - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\
&\quad - e^S (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha) \\
&\quad - e^S (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\gamma\beta} V^\beta + 2n^2 c V^\alpha H^\alpha H^\beta V^\beta) \\
&\quad - e^S (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + ncV^\alpha V^\alpha) dv.
\end{aligned}$$

**定理 9.20:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n, E)}$  超曲面,  $V = V^i e_i + fe_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
&= \int_M e^S (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} + 4nc f f_{,ij} h_{ij} H) \\
&\quad + e^S (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij,p} V^p + 2f f_{,ij} h_{ip} h_{pj} + 2cf \Delta f) \\
&\quad + e^S (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 + 4nc f^2 P_3 H) \\
&\quad + e^S [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3,i} V^i + 3P_4 f) \\
&\quad - 2f^2 (-c\delta_{ij} h_{jk} h_{ki} - c\delta_{jk} h_{ij} h_{ki} - c\delta_{ik} h_{ij} h_{jk})] \\
&\quad - e^S (-4ncH f h_{kl} f_{,kl} - 2nc f H S_{,p} V^p - 4nc f^2 H P_3 - 4n^2 c^2 f^2 H^2) \\
&\quad - e^S (-2cf \Delta f - 2nc f H_{,p} V^p - 2cf^2 S - 2c^2 n f^2) \\
&\quad - e^S (2nf H h_{kl} f_{,kl} + nf H S_{,p} V^p + 2nf^2 H P_3 + 2n^2 c f^2 H^2) \\
&\quad - e^S (f \Delta f + nf H_{,p} V^p + f^2 P_2 + nc f^2) dv.
\end{aligned}$$

**注释 9.3:** 特别注意上面诸定理之中黎曼张量  $\bar{R}_{i\alpha j\beta}$  分别在流形  $N$  和流形  $M$  上的拉回丛  $x^* TN$  上的两个协变导数  $\bar{R}_{i\alpha j\beta;p}$  和  $\bar{R}_{i\alpha j\beta, p}$  的区别。

## 9.4 $GD_{(n, \log)}$ 泛函的第二变分公式

我们知道当  $F(u) = \log u$ ,  $u > 0$  时, 泛函  $GD_{n, F}$  和  $GD_{(n, F, \epsilon)}$  分别变为

$$GD_{(n, \log)} = \int_M \log(S) dv, \quad GD_{(n, \log, \epsilon)} = \int_M \log(S + \epsilon) dv$$

对于  $GD_{(n, \log)}$  泛函, 我们显然要求其没有脐点。针对此类重要的特殊情形的计算, 可以直接利用的定理得到, 于是我们有以下定理。

**定理 9.21:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中无测地点的  $GD_{(n, \log)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
 & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
 = & \int_M \frac{-1}{S^2} (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^\rho V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
 & + \frac{1}{S} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^\rho + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
 & + \frac{-1}{S^2} (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^\rho + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
 & + \frac{1}{S} [2V^\alpha (V_{,ij}^\beta h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
 & + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
 & - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jky}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
 & + \frac{1}{S^2} (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^\rho + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
 & - \frac{1}{S} (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^\rho + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
 & - \frac{1}{S} (2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha\gamma}^\beta V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\alpha,p}^\beta V^\rho + 2V^\alpha h_{ij}^\beta \bar{R}_{\gamma\beta\alpha}^\beta V_{,j}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\alpha j\gamma}^\beta V_{,q}^\gamma \\
 & + 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta\gamma\alpha}^\beta V_{,j}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{i\beta j\gamma}^\beta V_{,q}^\gamma) \\
 & - \frac{1}{S} (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^\rho + 2nV^\alpha H^\alpha S_{\gamma\gamma\beta} - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
 & - \log(S) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^\rho + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dv.
 \end{aligned}$$

**定理 9.22:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n, \log, \epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
 & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\
 = & \int_M \frac{-1}{(S+\epsilon)^2} (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^\rho V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
 & + \frac{1}{S+\epsilon} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^\rho + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
 & + \frac{-1}{(S+\epsilon)^2} (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^\rho + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{S+\epsilon} [2V^\alpha (V_{,ij}^\alpha h_{ij}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta,i}^\alpha + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{ij\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
& + \frac{1}{(S+\epsilon)^2} (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\beta\beta\gamma} V^\gamma - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - \frac{1}{S+\epsilon} (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\gamma h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - \frac{1}{S+\epsilon} (2V^\alpha h_{ij}^\beta \bar{R}_{ij\beta\alpha;\gamma}^\beta V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\beta\alpha,p}^\beta V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\beta\alpha}^\beta V_{,i}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V_{,q}^\beta \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\beta\gamma\alpha}^\beta V_{,j}^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ij\beta\gamma}^\beta V_{,q}^\alpha) \\
& - \frac{1}{S+\epsilon} (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta\gamma} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& - \log(S+\epsilon) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\gamma V^\beta) dv.
\end{aligned}$$

**定理 9.23:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的无测地点  $GD_{(n, \log)}$  超曲面,

$V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
& = \int_M \frac{-1}{S^2} (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + \frac{1}{S} (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij,p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
& + \frac{-1}{S^2} (4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + \frac{1}{S} [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3,i} V^i + 3P_4 f) \\
& - 2^2 (\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
& + \frac{1}{S^2} (4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 \\
& - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) - \frac{1}{S} (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij,p} V^p \\
& + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
& - \frac{1}{S} (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1);(n+1)} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1),p} V^p + 2f h_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
& - 2f h_{ij} \bar{R}_{j(n+1)(n+1)} f_{,q} + 2f h_{ij} \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2f h_{ij} \bar{R}_{i(n+1)jq} f_{,q})
\end{aligned}$$



$$- \frac{1}{S} (2nfHh_{kl}f_{,kl} + 1fHS_{,p}V^p + 2nf^2HP_3 - 2nf^2Hh_{kl}\bar{R}_{k(n+1)l(n+1)})$$

$$- \log(S) (f\Delta f + nfH_{,p}V^p + f^2P_2 + f^2\bar{R}_{(n+1)ii(n+1)}) dv.$$

**定理 9.24:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n, \log, \epsilon)}$  超曲面,  $V = V^i e_i + fe_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned} & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\ = & \int_M \frac{-1}{(S + \epsilon)^2} (4h_{kl}f_{,kl}f_{,ij}h_{ij} + 2S_{,p}V^p f_{,ij}h_{ij} + 4P_3ff_{,ij}h_{ij} - 4ff_{,ij}h_{ij}h_{kl}\bar{R}_{k(n+1)l(n+1)}) \\ & + \frac{1}{S + \epsilon} (2f_{,ij}f_{,ij} + 2f_{,ij} + 2f_{,ij}h_{ij,p}V^p + 2ff_{,ij}h_{ip}h_{pj} - 2ff_{,ij}\bar{R}_{i(n+1)j(n+1)}) \\ & + \frac{-1}{(S + \epsilon)^2} (4fP_3h_{kl}f_{,kl} + 2fP_3S_{,p}V^p + 4f^2P_3P_3 - 4f^2P_3h_{kl}\bar{R}_{k(n+1)l(n+1)}) \\ & + \frac{1}{S + \epsilon} [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4f) \\ & - 2f^2\bar{R}_{i(n+1)j(n+1)}h_{jk}h_{ki} + h_{ij}\bar{R}_{j(n+1)k(n+1)}h_{ki} + h_{ij}h_{jk}\bar{R}_{k(n+1)i(n+1)}] \\ & + \frac{1}{(S + \epsilon)^2} (4fh_{ij}\bar{R}_{i(n+1)j(n+1)}h_{kl}f_{,kl} + 2fh_{ij}\bar{R}_{i(n+1)j(n+1)}S_{,p}V^p + 4f^2h_{ij}\bar{R}_{i(n+1)j(n+1)}P_3 \\ & - 4f^2h_{ij}\bar{R}_{i(n+1)j(n+1)}h_{kl}\bar{R}_{k(n+1)l(n+1)}) \\ & - \frac{1}{S + \epsilon} (2f\bar{R}_{i(n+1)j(n+1)}f_{,ij} + 2f\bar{R}_{i(n+1)j(n+1)}h_{ij,p}V^p \\ & + 2f^2\bar{R}_{i(n+1)j(n+1)}h_{ip}h_{pj} - 2f^2\bar{R}_{i(n+1)j(n+1)}\bar{R}_{i(n+1)j(n+1)}) \\ & - \frac{1}{S + \epsilon} (2f^2h_{ij}\bar{R}_{i(n+1)j(n+1);(n+1)} + 2fh_{ij}\bar{R}_{i(n+1)j(n+1),p}V^p + 2fh_{ij}\bar{R}_{(n+1)(n+1)j(n+1)}f_{,i} \\ & - 2fh_{ij}\bar{R}_{ij(n+1)}f_{,q} + 2fh_{ij}\bar{R}_{i(n+1)(n+1)(n+1)}f_{,i} - 2fh_{ij}\bar{R}_{i(n+1)jq}f_{,q}) \\ & - \frac{1}{S + \epsilon} (2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 - 2nf^2Hh_{kl}\bar{R}_{k(n+1)l(n+1)}) \\ & - \log(S + \epsilon) (f\Delta f + nfH_{,p}V^p + f^2P_2 + f^2\bar{R}_{(n+1)ii(n+1)}) dv. \end{aligned}$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\bar{R}_{ABCD} = -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^{\beta} = -c\delta_{ij}\delta_{\alpha\beta},$$

$$\bar{R}_{AB}^{\top} = \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c \sum_i \delta_{Ai}\delta_{iB},$$

$$\bar{R}_{AB}^{\perp} = \sum_{\alpha} \bar{R}_{A\alpha\alpha B} = \sum_{\alpha} -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c \sum_{\alpha} \delta_{A\alpha}\delta_{B\alpha},$$

$$\bar{R}_{\alpha\beta}^{\top} = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.$$

**定理 9.25:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的无测地点的  $GD_{(n, \log)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
 & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
 = & \int_M \frac{-1}{S^2} (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 4ncH^\beta V_{,ij}^\alpha h_{ij}^\alpha V^\beta) \\
 & + \frac{1}{S} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij, p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha) V^\alpha) \\
 & + \frac{-1}{S^2} (4V^\alpha S_{\alpha\beta\gamma} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\gamma} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\gamma} S_{\delta\gamma} V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta\gamma} V^\gamma) \\
 & + \frac{1}{S} [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
 & + 2V^\alpha (S_{\alpha\beta\gamma, i} V^i + S_{\alpha\gamma\beta\delta} V^\delta + S_{\alpha\beta\gamma\delta} V^\delta + S_{\alpha\beta\gamma\delta} V^\delta) - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta})] \\
 & + \frac{1}{S^2} (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\gamma} V^\gamma - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\
 & - \frac{1}{S} (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha) \\
 & - \frac{1}{S} (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta + 2n^2 cV^\alpha H^\alpha H^\beta V^\beta) \\
 & - \log(S) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + ncV^\alpha V^\alpha) dv.
 \end{aligned}$$

**定理 9.26:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n, \log, \epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
 & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\
 = & \int_M \frac{-1}{(S + \epsilon)^2} (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 4ncH^\beta V_{,ij}^\alpha h_{ij}^\alpha V^\beta) \\
 & + \frac{1}{S + \epsilon} (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij, p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha) V^\alpha) \\
 & + \frac{-1}{(S + \epsilon)^2} (4V^\alpha S_{\alpha\beta\gamma} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\gamma} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\gamma} S_{\delta\gamma} V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta\gamma} V^\gamma) \\
 & + \frac{1}{S + \epsilon} [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
 & + 2V^\alpha (S_{\alpha\beta\gamma, i} V^i + S_{\alpha\gamma\beta\delta} V^\delta + S_{\alpha\beta\gamma\delta} V^\delta + S_{\alpha\beta\gamma\delta} V^\delta) - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta})] \\
 & + \frac{1}{(S + \epsilon)^2} (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\gamma} V^\gamma - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\
 & - \frac{1}{S + \epsilon} (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{S+\epsilon}(2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p}V^p + 2nV^\alpha H^\alpha S_{\gamma\beta}V^\beta + 2n^2cV^\alpha H^\alpha H^\beta V^\beta) \\
& -\log(S+\epsilon)(V^\alpha\Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta}V^\beta + ncV^\alpha V^\alpha)dv.
\end{aligned}$$

**定理 9.27:** 假设  $x:M^n \rightarrow R^{n+1}(c)$  是空间形式之中无测地点的  $GD_{(n, \log)}$  超曲面,  $V = V^i e_i + fe_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, \log)}(x_t) \\
& = \int_M \frac{-1}{S^2} (4h_{kl}f_{,kl}f_{,ij}h_{ij} + 2S_{,p}V^p f_{,ij}h_{ij} + 4P_3 f_{,ij}f_{,ij}h_{ij} + 4nc f_{,ij}f_{,ij}h_{ij}H) \\
& + \frac{1}{S} (2f_{,ij}f_{,ij} + 2f_{,ij}h_{ij,p}V^p + 2f_{,ij}h_{ip}h_{pj} + 2cf\Delta f) \\
& + \frac{-1}{S^2} (4fP_3 h_{kl}f_{,kl} + 2fP_3 S_{,p}V^p + 4f^2 P_3 P_3 + 4ncf^2 P_3 H) \\
& + \frac{1}{S} [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4 f) \\
& - 2f^2(-c\delta_{ij}h_{jk}h_{ki} - c\delta_{jk}h_{ij}h_{ki} - c\delta_{ik}h_{ij}h_{jk})] \\
& + \frac{1}{S^2} (-4ncHf_{kl}f_{,kl} - 2ncfHS_{,p}V^p - 4ncf^2 HP_3 - 4n^2c^2f^2H^2) \\
& - \frac{1}{S} (-2cf\Delta f - 2ncfH_{,p}V^p - 2cf^2S - 2c^2nf^2) \\
& - \frac{1}{S} (2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 + 2n^2cf^2H^2) \\
& - \log(S)(f\Delta f + nfH_{,p}V^p + f^2P_2 + nc f^2)dv.
\end{aligned}$$

**定理 9.28:** 假设  $x:M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n, \log, \epsilon)}$  超曲面,  $V = V^i e_i + fe_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, \log, \epsilon)}(x_t) \\
& = \int_M \frac{-1}{(S+\epsilon)^2} (4h_{kl}f_{,kl}f_{,ij}h_{ij} + 2S_{,p}V^p f_{,ij}h_{ij} + 4P_3 f_{,ij}f_{,ij}h_{ij} + 4nc f_{,ij}f_{,ij}h_{ij}H) \\
& + \frac{1}{S+\epsilon} (2f_{,ij}f_{,ij} + 2f_{,ij}h_{ij,p}V^p + 2f_{,ij}h_{ip}h_{pj} + 2cf\Delta f) \\
& + \frac{-1}{(S+\epsilon)^2} (4fP_3 h_{kl}f_{,kl} + 2fP_3 S_{,p}V^p + 4f^2 P_3 P_3 + 4ncf^2 P_3 H) \\
& + \frac{1}{S+\epsilon} [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4 f) \\
& - 2f^2(-c\delta_{ij}h_{jk}h_{ki} - c\delta_{jk}h_{ij}h_{ki} - c\delta_{ik}h_{ij}h_{jk})]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(S + \epsilon)^2} (-4ncHf h_{kl} f_{,kl} - 2ncfHS_{,p} V^p - 4ncf^2 HP_3 - 4n^2 c^2 f^2 H^2) \\
& - \frac{1}{S + \epsilon} (-2cf\Delta f - 2ncfH_{,p} V^p - 2cf^2 S - 2c^2 nf^2) \\
& - \frac{1}{S + \epsilon} (2nfHh_{kl} f_{,kl} + nfHS_{,p} V^p + 2nf^2 HP_3 + 2n^2 cf^2 H^2) \\
& - \log(S + \epsilon) (f\Delta f + nfH_{,p} V^p + f^2 P_2 + nc f^2) dv.
\end{aligned}$$

**注释 9.4:** 特别注意上面诸定理之中黎曼张量  $\bar{R}_{i\alpha\beta}$  分别在流形  $N$  和流形  $M$  上的拉回丛  $x^*TN$  上的两个协变导数  $\bar{R}_{i\alpha\beta,p}$  和  $\bar{R}_{i\alpha\beta,p}$  的区别。

## 9.5 $GD_{(n, \sin)}$ 泛函的第二变分公式

我们知道当  $F(u) = \sin(u)$  时, 泛函  $GD_{n, F}$  和  $GD_{(n, F, \epsilon)}$  分别变为

$$GD_{(n, \sin)} = \int_M \sin(S) dv, \quad GD_{(n, \sin, \epsilon)} = \int_M \sin(S + \epsilon) dv$$

针对此类重要的特殊情形的计算, 可以直接利用上面一节的定理得到, 于是我们有

**定理 9.29:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n, \sin)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_i) \\
& = \int_M -\sin(S) (4h_{kl}^\beta V_{kl}^\alpha V_{ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V_{ij}^\alpha h_{ij}^\alpha - 4V_{ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + \cos(S) (2V_{ij}^\alpha V_{ij}^\alpha + 2V_{ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
& - \sin(S) (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& + \cos(S) (2V^\alpha (V_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma) \\
& + \sin(S) (4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{kl}^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\gamma} V^\gamma \\
& - 4V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - \cos(S) (2V^\alpha \bar{R}_{ij\alpha}^\beta V_{ij}^\beta + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\gamma h_{pj}^\gamma V^\gamma - 2V^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - \cos(S) (2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha\gamma}^\beta V^\gamma + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha,p}^\beta V^p + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V_{\gamma\beta}^\gamma V^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta V_{\gamma\beta}^\gamma V^\gamma \\
& + 2V^\alpha h_{ij}^\beta \bar{R}_{ij\gamma\alpha}^\beta V_j^\gamma - 2V^\alpha h_{ij}^\beta \bar{R}_{ij\gamma q}^\beta V_q^\gamma)
\end{aligned}$$

$$\begin{aligned}
& -\cos(S)(2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& -\sin(S)(V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\top V^\beta) dv.
\end{aligned}$$

**定理 9.30:** 假设  $x: M^n \rightarrow N^{n+p}$  是一般原流形之中的  $GD_{(n, \sin, \epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD(n, F, \epsilon)(x_t) \\
= & \int_M -\sin(S + \epsilon)(4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha - 4V_{,ij}^\alpha h_{ij}^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& + \cos(S + \epsilon)(2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij,p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta - 2V_{,ij}^\alpha \bar{R}_{ij\beta}^\alpha V^\beta) \\
& - \sin(S + \epsilon)(4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma - 4V^\alpha S_{\alpha\beta\beta} h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& + \cos(S + \epsilon)[2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\
& + 2V^\alpha (S_{\alpha\beta\beta,i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) \\
& - 2V^\alpha (\bar{R}_{ij\gamma}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha \bar{R}_{jk\gamma}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta \bar{R}_{ki\gamma}^\beta) V^\gamma] \\
& + \sin(S + \epsilon)(4V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\gamma V_{,kl}^\gamma + 2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{,p} V^p + 4V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta S_{\delta\delta\gamma} V^\gamma \\
& - 4V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\alpha}^\beta h_{kl}^\delta \bar{R}_{kl\gamma}^\delta V^\gamma) \\
& - \cos(S + \epsilon)(2V_{,ij}^\alpha \bar{R}_{ij\alpha}^\beta V_{,ij}^\beta + 2V_{,ij}^\alpha \bar{R}_{ij\alpha}^\beta h_{ij,p}^\beta V^p + 2V_{,ij}^\alpha \bar{R}_{ij\alpha}^\beta h_{ip}^\beta h_{pj}^\gamma V^\gamma - 2V_{,ij}^\alpha \bar{R}_{ij\alpha}^\beta \bar{R}_{ij\gamma}^\beta V^\gamma) \\
& - \cos(S + \epsilon)(2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\beta\alpha,\gamma} V^\gamma + 2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\beta\alpha,p} V^p + 2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\beta\alpha} V_{,i}^\gamma - 2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{iq,j\alpha} V_{,q}^\beta \\
& + 2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\beta\gamma\alpha} V_{,j}^\gamma - 2V_{,ij}^\alpha h_{ij}^\beta \bar{R}_{ij\beta q} V_{,q}^\alpha) \\
& - \cos(S + \epsilon)(2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta - 2nV^\alpha H^\alpha h_{kl}^\gamma \bar{R}_{kl\beta}^\gamma V^\beta) \\
& - \sin(S + \epsilon)(V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + V^\alpha \bar{R}_{\alpha\beta}^\top V^\beta) dv.
\end{aligned}$$

**定理 9.31:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n, \sin)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\
= & \int_M -\sin(S)(4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + \cos(S)(2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij,p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
& - \sin(S)(4f P_3 h_{kl} f_{,kl} + 2f P_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + \cos(S)[2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_3, i V^i + 3P_4 f) \\
& - 2f^2(\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
& + \sin(S)(4f h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2f h_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3
\end{aligned}$$

$$\begin{aligned}
& -4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& -\cos(S) (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij, p} V^p \\
& + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
& -\cos(S) (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1);(n+1)} + 2fh_{ij} \bar{R}_{i(n+1)j(n+1), p} V^p + 2fh_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
& - 2fh_{ij} \bar{R}_{iqj(n+1)} f_{,q} + 2fh_{ij}^B \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2fh_{ij} \bar{R}_{i(n+1)jq} f_{,q}) \\
& -\cos(S) (2nfHh_{kl} f_{,kl} + nfHS_{,p} V^p + 2nf^2 HP_3 - 2nf^2 Hh_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& -\sin(S) (f\Delta f + nfH_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)\dot{u}(n+1)}) dv.
\end{aligned}$$

**定理 9.32:** 假设  $x: M^n \rightarrow N^{n+1}$  是一般原流形之中的  $GD_{(n, \sin, \epsilon)}$  超曲面,  $V = V^i e_i + fe_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\
& = \int_M -\sin(S + \epsilon) (4h_{kl} f_{,kl} f_{,ij} h_{ij} + 2S_{,p} V^p f_{,ij} h_{ij} + 4P_3 f f_{,ij} h_{ij} - 4f f_{,ij} h_{ij} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + \cos(S + \epsilon) (2f_{,ij} f_{,ij} + 2f_{,ij} h_{ij, p} V^p + 2f f_{,ij} h_{ip} h_{pj} - 2f f_{,ij} \bar{R}_{i(n+1)j(n+1)}) \\
& - \sin(S + \epsilon) (4fP_3 h_{kl} f_{,kl} + 2fP_3 S_{,p} V^p + 4f^2 P_3 P_3 - 4f^2 P_3 h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& + \cos(S + \epsilon) [2f(f_{,ij} h_{jk} h_{ki} + h_{ij} f_{,jk} h_{ki} + h_{ij} h_{jk} f_{,ki}) + 2f(P_{3,i} V^i + 3P_4 f) \\
& - 2f^2 (\bar{R}_{i(n+1)j(n+1)} h_{jk} h_{ki} + h_{ij} \bar{R}_{j(n+1)k(n+1)} h_{ki} + h_{ij} h_{jk} \bar{R}_{k(n+1)i(n+1)})] \\
& + \sin(S + \epsilon) (4fh_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} f_{,kl} + 2fh_{ij} \bar{R}_{i(n+1)j(n+1)} S_{,p} V^p + 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} P_3 \\
& - 4f^2 h_{ij} \bar{R}_{i(n+1)j(n+1)} h_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& - \cos(S + \epsilon) (2f \bar{R}_{i(n+1)j(n+1)} f_{,ij} + 2f \bar{R}_{i(n+1)j(n+1)} h_{ij, p} V^p \\
& + 2f^2 \bar{R}_{i(n+1)j(n+1)} h_{ip} h_{pj} - 2f^2 \bar{R}_{i(n+1)j(n+1)} \bar{R}_{i(n+1)j(n+1)}) \\
& - \cos(S + \epsilon) (2f^2 h_{ij} \bar{R}_{i(n+1)j(n+1);(n+1)} + 2fh_{ij} \bar{R}_{i(n+1)j(n+1), p} V^p + 2fh_{ij} \bar{R}_{(n+1)(n+1)j(n+1)} f_{,i} \\
& - 2fh_{ij} \bar{R}_{iqj(n+1)} f_{,q} + 2fh_{ij}^B \bar{R}_{i(n+1)(n+1)(n+1)} f_{,j} - 2fh_{ij} \bar{R}_{i(n+1)jq} f_{,q}) \\
& - \cos(S + \epsilon) (2nfHh_{kl} f_{,kl} + nfHS_{,p} V^p + 2nf^2 HP_3 - 2nf^2 Hh_{kl} \bar{R}_{k(n+1)l(n+1)}) \\
& - \sin(S + \epsilon) (f\Delta f + nfH_{,p} V^p + f^2 P_2 + f^2 \bar{R}_{(n+1)\dot{u}(n+1)}) dv.
\end{aligned}$$

当流形  $N^{n+p}$  是空间形式  $R^{n+p}(c)$  时, 我们知道其黎曼曲率张量可以表达为

$$\begin{aligned}
\bar{R}_{ABCD} &= -c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \quad \bar{R}_{ij\alpha}^B = -c\delta_{ij}\delta_{\alpha\beta}, \\
\bar{R}_{AB}^T &= \sum_i \bar{R}_{AiiB} = \sum_i -c(\delta_{Ai}\delta_{iB} - \delta_{AB}\delta_{ii}) = nc\delta_{AB} - c \sum_i \delta_{Ai}\delta_{iB}, \\
\bar{R}_{AB}^\perp &= \sum_\alpha \bar{R}_{A\alpha\alpha B} = \sum_\alpha -c(\delta_{A\alpha}\delta_{\alpha B} - \delta_{AB}\delta_{\alpha\alpha}) = pc\delta_{AB} - c \sum_\alpha \delta_{A\alpha}\delta_{B\alpha},
\end{aligned}$$

$$\bar{R}_{\alpha\beta}^{\top} = nc\delta_{\alpha\beta}, \quad \bar{R}_{ij}^{\perp} = pc\delta_{ij}.$$

**定理 9.33:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n, \sin)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned} & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) \\ &= \int_M -\sin(S) (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 4ncH^\beta V_{,ij}^\alpha h_{ij}^\alpha V^\beta) \\ & \quad + \cos(S) (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij, p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha) V^\alpha) \\ & \quad - \sin(S) (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta\beta} V^\gamma) \\ & \quad + \cos(S) [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\ & \quad + 2V^\alpha (S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta})] \\ & \quad + \sin(S) (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\delta\gamma} V^\gamma - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\ & \quad - \cos(S) (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha) \\ & \quad - \cos(S) (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta + 2n^2 c V^\alpha H^\alpha H^\beta V^\beta) \\ & \quad - \sin(S) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + ncV^\alpha V^\alpha) dv. \end{aligned}$$

**定理 9.34:** 假设  $x: M^n \rightarrow R^{n+p}(c)$  是空间形式之中的  $GD_{(n, \sin, \epsilon)}$  子流形,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned} & \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_t) \\ &= \int_M -\sin(S + \epsilon) (4h_{kl}^\beta V_{,kl}^\beta V_{,ij}^\alpha h_{ij}^\alpha + 2S_{,p} V^p V_{,ij}^\alpha h_{ij}^\alpha + 4S_{\gamma\beta} V^\beta V_{,ij}^\alpha h_{ij}^\alpha + 4ncH^\beta V_{,ij}^\alpha h_{ij}^\alpha V^\beta) \\ & \quad + \cos(S + \epsilon) (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ij, p}^\alpha V^p + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 2c\Delta(V^\alpha) V^\alpha) \\ & \quad - \sin(S + \epsilon) (4V^\alpha S_{\alpha\beta\beta} h_{kl}^\gamma V_{,kl}^\gamma + 2V^\alpha S_{\alpha\beta\beta} S_{,p} V^p + 4V^\alpha S_{\alpha\beta\beta} S_{\delta\delta\gamma} V^\gamma + 4ncH^\gamma V^\alpha S_{\alpha\beta\beta} V^\gamma) \\ & \quad + \cos(S + \epsilon) [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\ & \quad + 2V^\alpha (S_{\alpha\beta\beta, i} V^i + S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma) - 2V^\alpha (-cV^\alpha S - 2cV^\beta S_{\alpha\beta})] \\ & \quad + \sin(S + \epsilon) (-4ncV^\alpha H^\alpha h_{kl}^\gamma V_{,kl}^\gamma - 2ncV^\alpha H^\alpha S_{,p} V^p - 4ncV^\alpha H^\alpha S_{\delta\delta\gamma} V^\gamma \\ & \quad - 4n^2 c^2 V^\alpha H^\alpha H^\gamma V^\gamma) \\ & \quad - \cos(S + \epsilon) (-2cV^\alpha \Delta(V^\alpha) - 2ncV^\alpha H_{,p}^\alpha V^p - 2cV^\alpha S_{\alpha\gamma} V^\gamma - 2nc^2 V^\alpha V^\alpha) \\ & \quad - \cos(S + \epsilon) (2nV^\alpha H^\alpha h_{kl}^\beta V_{,kl}^\beta + nV^\alpha H^\alpha S_{,p} V^p + 2nV^\alpha H^\alpha S_{\gamma\beta} V^\beta + 2n^2 c V^\alpha H^\alpha H^\beta V^\beta) \\ & \quad - \sin(S + \epsilon) (V^\alpha \Delta V^\alpha + nV^\alpha H_{,p}^\alpha V^p + V^\alpha S_{\alpha\beta} V^\beta + ncV^\alpha V^\alpha) dv. \end{aligned}$$

**定理 9.35:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n, \sin)}$  超曲面,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_i) \\
= & \int_M -\sin(S) (4h_{kl}f_{,kl}f_{,ij}h_{ij} + 2S_{,p}V^pf_{,ij}h_{ij} + 4P_3ff_{,ij}h_{ij} + 4ncff_{,ij}h_{ij}H) \\
& + \cos(S) (2f_{,ij}f_{,ij} + 2f_{,ij}h_{ij,p}V^p + 2ff_{,ij}h_{ip}h_{pj} + 2cf\Delta f) \\
& - \sin(S) (4fP_3h_{kl}f_{,kl} + 2fP_3S_{,p}V^p + 4f^2P_3P_3 + 4ncf^2P_3H) \\
& + \cos(S) [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4f) \\
& - 2f^2(-c\delta_{ij}h_{jk}h_{ki} - c\delta_{jk}h_{ij}h_{ki} - c\delta_{ik}h_{ij}h_{jk})] \\
& + \sin(S) (-4ncHf_{kl}f_{,kl} - 2ncfHS_{,p}V^p - 4ncf^2HP_3 - 4n^2c^2f^2H^2) \\
& - \cos(S) (-2cf\Delta f - 2ncfH_{,p}V^p - 2cf^2S - 2c^2nf^2) \\
& - \cos(S) (2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 + 2n^2cf^2H^2) \\
& - \sin(S) (f\Delta f + nfH_{,p}V^p + f^2P_2 + nc f^2) dv.
\end{aligned}$$

**定理 9.36:** 假设  $x: M^n \rightarrow R^{n+1}(c)$  是空间形式之中的  $GD_{(n, \sin, \epsilon)}$  超曲面,  $V = Ve_i + fe_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F, \epsilon)}(x_i) \\
= & \int_M -\sin(S + \epsilon) (4h_{kl}f_{,kl}f_{,ij}h_{ij} + 2S_{,p}V^pf_{,ij}h_{ij} + 4P_3ff_{,ij}h_{ij} + 4ncff_{,ij}h_{ij}H) \\
& + \cos(S + \epsilon) (2f_{,ij}f_{,ij} + 2f_{,ij}h_{ij,p}V^p + 2ff_{,ij}h_{ip}h_{pj} + 2cf\Delta f) \\
& - \sin(S + \epsilon) (4fP_3h_{kl}f_{,kl} + 2fP_3S_{,p}V^p + 4f^2P_3P_3 + 4ncf^2P_3H) \\
& + \cos(S + \epsilon) [2f(f_{,ij}h_{jk}h_{ki} + h_{ij}f_{,jk}h_{ki} + h_{ij}h_{jk}f_{,ki}) + 2f(P_{3,i}V^i + 3P_4f) \\
& - 2f^2(-c\delta_{ij}h_{jk}h_{ki} - c\delta_{jk}h_{ij}h_{ki} - c\delta_{ik}h_{ij}h_{jk})] \\
& + \sin(S + \epsilon) (-4ncHf_{kl}f_{,kl} - 2ncfHS_{,p}V^p - 4ncf^2HP_3 - 4n^2c^2f^2H^2) \\
& - \cos(S + \epsilon) (-2cf\Delta f - 2ncfH_{,p}V^p - 2cf^2S - 2c^2nf^2) \\
& - \cos(S + \epsilon) (2nfHh_{kl}f_{,kl} + nfHS_{,p}V^p + 2nf^2HP_3 + 2n^2cf^2H^2) \\
& - \sin(S + \epsilon) (f\Delta f + nfH_{,p}V^p + f^2P_2 + nc f^2) dv.
\end{aligned}$$

**注释 9.5:** 特别注意上面诸定理之中黎曼张量  $\bar{R}_{\alpha\beta\gamma}$  分别在流形  $N$  和流形  $M$  上的拉回丛  $x^*TN$  上的两个协变导数  $\bar{R}_{\alpha\beta\gamma,p}$  和  $\bar{R}_{\alpha\beta\gamma}$  的区别。

## 9.6 单位球面中 $GD_{(n, F)}$ 子流形的稳定性

在本节, 我们讨论单位球面之中的  $GD_{(n, F)}$  子流形的稳定性, 我们考虑三个典型例子——全测地超曲面、Clifford 超曲面  $C_{\frac{n}{2}, \frac{n}{2}}$  和 Veronese 曲面。

**例 9.1:** 全测地超曲面按照其定义, 我们知道所有的主曲率为



$$k_1 = k_2 = \cdots = 0.$$

于是, 可以计算为

$$P_1 = 0, P_2 = 0, P_3 = 0.$$

代入上面的方程, 我们可以做结论为对于任意的参数函数  $F \in C^3[0, \infty)$ , 全测地超曲面  $M$  为  $GD_{(n, F)}$  超曲面。

**例 9.2:** 对于维数为偶数  $n \equiv 0 \pmod{2}$  的特殊 Clifford 超曲面

$$C_{\frac{n}{2}, \frac{n}{2}} = S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \times S^{\frac{n}{2}}\left(\frac{1}{\sqrt{2}}\right) \rightarrow S^{n+1}(1).$$

我们知道所有的主曲率为

$$k_1 = \cdots = k_{\frac{n}{2}} = 1, k_{\frac{n}{2}+1} = \cdots = k_n = -1.$$

于是可以计算所有的曲率函数  $P_1, P_2, P_3$  为

$$P_1 = 0, P_2 = n, P_3 = 0.$$

于是我们得到  $C_{\frac{n}{2}, \frac{n}{2}}$  对于任何函数  $F \in C^3(0, \infty)$  都是  $GD_{(n, F)}$  超曲面。

**例 9.3:** 假设  $(x, y, z)$  是三维欧式空间  $R^3$  的自然标架, 假设  $(u_1, u_2, u_3, u_4, u_5)$  是五维欧式空间  $R^5$  的自然标架, 我们定义如下的映射为

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{3}}yz, u_2 = \frac{1}{\sqrt{3}}xz, u_3 = \frac{1}{\sqrt{3}}xy, \\ u_4 &= \frac{1}{2\sqrt{3}}(x^2 - y^2), u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2), \\ x^2 + y^2 + z^2 &= 3 \end{aligned}$$

这个映射决定了一个等距嵌入  $x: RP^2 = S^2(\sqrt{3})/Z_2 \rightarrow S^4(1)$ , 我们称其为 Veronese 曲面, 通过简单的计算, 我们知道第二基本型为

$$A_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, A_4 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

通过上面的第二基本型和定义, 我们可以计算得到

$$H^3 = H^4 = 0, S_{33} = S_{44} = \frac{2}{3}, S = \rho = \frac{4}{3},$$

$$S_{34} = S_{43} = 0, S_{333} = S_{344} = S_{433} = S_{444} = 0.$$

显然, Veronese 曲面对于任意的函数  $F \in C^3(0, \infty)$  都是  $GD_{(2, F)}$  曲面。

**定理 9.37:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  是单位球面之中的  $GD_{(n, F)}$  超曲面且为全测地的,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) = \int_M F'(0) (2f_{,ij} f_{,ij} + 4f \Delta f + 2n f^2) - F(0) (f \Delta f + n f^2) dv.$$

**定理 9.38:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  ( $n$  是偶数) 是单位球面之中的  $GD_{(n, F)}$  超曲面且为  $C_{\frac{n}{2}, \frac{n}{2}}$ ,  $V = V^i e_i + f e_{n+1}$  是变分向量场, 那么其第二变分为

$$\begin{aligned} \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(n, F)}(x_t) &= \int_M 4F''(n) \left( \sum_{i=1}^{\frac{n}{2}} f_{,ii} - \sum_{j=\frac{n}{2}+1}^n f_{,jj} \right)^2 \\ &\quad + F'(n) (2f_{,ij} f_{,ij} + 12f \Delta f + 16nf^2) - F(n) (f \Delta f + 2nf^2) \, dv. \end{aligned}$$

**定理 9.39:** 假设  $x: M^2 \rightarrow S^4(1)$  是单位球面之中的  $GD_{(2, F)}$  子流形且为 Veronese 曲面,  $V = V^i e_i + V^\alpha e_\alpha$  是变分向量场, 那么其第二变分为

$$\begin{aligned} \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} GD_{(2, F)}(x_t) &= \int_M F''\left(\frac{4}{3}\right) (4h_{kl}^\beta V_{,kl}^\alpha V_{,ij}^\alpha h_{ij}^\alpha) \\ &\quad + F'\left(\frac{4}{3}\right) (2V_{,ij}^\alpha V_{,ij}^\alpha + 2V_{,ij}^\alpha h_{ip}^\alpha h_{pj}^\beta V^\beta + 4\Delta(V^\alpha) V^\alpha) \\ &\quad + F'\left(\frac{4}{3}\right) [2V^\alpha (V_{,ij}^\alpha h_{jk}^\beta h_{ki}^\beta + h_{ij}^\alpha V_{,jk}^\beta h_{ki}^\beta + h_{ij}^\alpha h_{jk}^\beta V_{,ki}^\beta) \\ &\quad + 2V^\alpha (S_{\alpha\gamma\beta\beta} V^\gamma + S_{\alpha\beta\gamma\beta} V^\gamma + S_{\alpha\beta\beta\gamma} V^\gamma)] \\ &\quad + F'\left(\frac{4}{3}\right) (6S_{\alpha\gamma} V^\alpha V^\gamma + \frac{20}{3} |V^\perp|^2) \\ &\quad - F\left(\frac{4}{3}\right) (V^\alpha \Delta V^\alpha + S_{\alpha\beta} V^\alpha V^\beta + 2|V^\perp|^2) \, dv. \end{aligned}$$

## 第 10 章 Simons 型积分不等式

子流形理论之中, Simons 类型积分不等式是一种重要的积分不等式, 其是讨论子流形间隙现象的基础。

### 10.1 矩阵不等式

Simons 型积分不等式是子流形几何中的一类重要的积分不等式, 在子流形间隙现象的研究和刚性定理的发展之中具有重要作用。实际上, 最原始的 Simons 型积分不等式是针对极小子流形推导出来的, 后来发现不仅在极小子流形之中有此现象, 而且在 Willomre 型泛函和子流形之中也有此类现象。本节的主要目的是研究  $GD_{(n, F)}$  和  $GD_{(n, F, \epsilon)}$  子流形的积分不等式。为此, 我们需要几个引理。

**引理 10.1:**

$$\begin{aligned} N(A_\alpha) &= \sum_{ij} (h_{ij}^\alpha)^2 =: S_{\alpha\alpha}, \quad N(\hat{A}_\alpha) = \sum_{ij} (\hat{h}_{ij}^\alpha)^2 = \hat{S}_{\alpha\alpha} = S_{\alpha\alpha} - n(H^\alpha)^2 \\ N(A_\alpha A_\beta - A_\beta A_\alpha) &= N((\hat{A}_\alpha + H^\alpha)A_\beta - A_\beta(\hat{A}_\alpha + H^\alpha)) = N(\hat{A}_\alpha A_\beta - A_\beta \hat{A}_\alpha) \\ &= N(\hat{A}_\alpha(\hat{A}_\beta + H^\beta) - (\hat{A}_\beta + H^\beta)\hat{A}_\alpha) = N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha). \end{aligned} \quad (10-1)$$

陈省身等人证明了下面的重要不等式, 为方便起见可以称为陈省身类型不等式。

**引理 10.2** (参见文献[3]): 设  $A, B$  是对称方阵, 那么

$$N(AB - BA) \leq 2N(A)N(B)$$

等式成立当且仅当两种情形, 情形 1:  $A, B$  至少有一个为 0; 情形 2: 如果  $A \neq 0, B \neq 0$ , 那么  $A, B$  可以同时正交化为下面的矩阵

$$A = \lambda \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \mu \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

如果  $B_1, B_2, B_3$  为对称阵且满足

$$N(B_i B_j - B_j B_i) = 2N(B_i)N(B_j), \quad 1 \leq i, j \leq 3$$

那么至少有一个为 0。李安民等人细致研究上面的陈省身类型不等式, 给出了更加精

细的不等式和等式成立的条件, 为了方便起见, 我们可以称之为李安民类型不等式。

**引理 10.3:** 假设  $A_1, A_2, \dots, A_p, p \geq 2$  是对称的  $(n \times n)$  矩阵, 令

$$S_{\alpha\beta} = \text{tr}(A_\alpha A_\beta), S_{\alpha\alpha} = N(A_\alpha), S = \sum_{\alpha} S_{\alpha\alpha}$$

我们有不等式

$$\sum_{\alpha \neq \beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha\beta} (S_{\alpha\beta})^2 \leq \frac{3}{2} S^2$$

等式成立当且仅当下面的情形之一成立

**情形 1:**  $A_1 = A_2 = \dots = A_p = 0$ .

**情形 2:**  $A_1 \neq 0, A_2 \neq 0, A_3 = A_4 = \dots = A_p = 0, S_{11} = S_{22}$ .

并且在情形 2 的条件之下,  $A_1, A_2$  可以同时正交化为下面的矩阵

$$A_1 = \sqrt{\frac{S_{11}}{2}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_2 = \sqrt{\frac{S_{22}}{2}} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

下面的张量不等式首先是由 Huisken 在超曲面的情形发现的, 在积分估计之中有重大应用。

**引理 10.4:** Huisken 的估计 (参见文献[93])。

• 余维数为 1 时

$$|\nabla h|^2 \geq \frac{3n^2}{n+2} |\nabla H|^2 \geq n |\nabla H|^2,$$

并且  $|\nabla h|^2 = n |\nabla H|^2$  当且仅当  $\nabla h = 0$ .

• 余维数大于等于 2 时

$$|\nabla h|^2 \geq \frac{3n^2}{n+2} |\nabla \vec{H}|^2 \geq n |\nabla \vec{H}|^2,$$

并且  $|\nabla h|^2 = n |\nabla \vec{H}|^2$  当且仅当  $\nabla h = 0$ .

**证明:** 分解张量  $h_{ij}^\alpha$  为

$$h_{ij,k}^\alpha = E_{ijk}^\alpha + F_{ijk}^\alpha$$

其中

$$E_{ijk}^\alpha = \frac{n}{n+2} (H_{,i}^\alpha \delta_{jk} + H_{,j}^\alpha \delta_{ik} + H_{,k}^\alpha \delta_{ij}), F_{ij}^\alpha = h_{ij}^\alpha - E_{ij}^\alpha.$$

直接计算

$$|E|^2 = \frac{3n^2}{n+2} |\nabla \vec{H}|^2, E \cdot F = 0,$$

那么利用三角不等式可得

$$|\nabla h|^2 \geq |E|^2 = \frac{3n^2}{n+2} |\nabla \vec{H}|^2 \geq n |\nabla \vec{H}|^2.$$

当

$$\nabla h = 0$$

时, 上面不等式之中的各项全部变成 0, 显然

$$|\nabla h|^2 = n |\nabla \vec{H}|^2.$$

反过来, 当

$$|\nabla h|^2 = n |\nabla \vec{H}|^2$$

时, 上面不等式之中的不等号全部变成等号, 于是

$$F_{ijk}^\alpha = 0, E_{ijk}^\alpha = 0, h_{ij,k}^\alpha = 0.$$

即是

$$\nabla h = 0.$$

综上

$$|\nabla h|^2 = n |\nabla \vec{H}|^2$$

当且仅当

$$\nabla h = 0.$$

Simons 积分不等式的推导依赖对曲率模长协变导数, 为此我们在各种情况之下详细计算了曲率模长的二阶协变导数。

**引理 10.5:** 对于曲率模长  $S$ , 其协变导数是

• 在一般流形之中且  $p \geq 2$  时

$$\begin{aligned} S_{,kl} &= \sum_{ij\alpha} -2h_{ij}^\alpha \bar{R}_{ijk,l}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha \bar{R}_{kli,j}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha h_{kl,i}^\alpha + \sum_{ij\alpha} 2h_{ij,k}^\alpha h_{ij,l}^\alpha \\ &\quad + 2 \left[ \sum_{ijp\alpha} h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjl}^\alpha + \sum_{ijp\alpha} h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjl}^\alpha + \sum_{ij\alpha\beta} h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jl}^\alpha \right. \\ &\quad + \sum_{ijp\alpha\beta} (h_{ij}^\alpha h_{il}^\beta h_{kp}^\alpha h_{pj}^\beta - h_{ij}^\alpha h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta) + \sum_{ijp\alpha\beta} (h_{ij}^\alpha h_{ip}^\alpha h_{pj}^\beta h_{kl}^\beta - h_{ij}^\alpha h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) \\ &\quad \left. + \sum_{ijp\alpha\beta} (h_{ij}^\alpha h_{ik}^\beta h_{jp}^\beta h_{pl}^\alpha - h_{ij}^\alpha h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta) \right], \\ \Delta S &= \sum_{ijk\alpha} -2h_{ij}^\alpha \bar{R}_{ijk,k}^\alpha + \sum_{ijk\alpha} 2h_{ij}^\alpha \bar{R}_{kki,j}^\alpha + \sum_{ij\alpha} 2nh_{ij}^\alpha H_{,ij}^\alpha + 2|Dh|^2 \\ &\quad + 2 \left[ \sum_{ijp\alpha} h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk}^\alpha + \sum_{ijp\alpha} h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk}^\alpha + \sum_{ij\alpha\beta} h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}^\alpha \right] \\ &\quad + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta\beta} H^\beta - 2 \sum_{\alpha\beta} [N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2] \\ &= \sum_{ijk\alpha} -2h_{ij}^\alpha \bar{R}_{ijk,k}^\alpha + \sum_{ijk\alpha} 2h_{ij}^\alpha \bar{R}_{kki,j}^\alpha + \sum_{ij\alpha} 2nh_{ij}^\alpha H_{,ij}^\alpha + 2|Dh|^2 \\ &\quad + 2 \left[ \sum_{ijp\alpha} h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk}^\alpha + \sum_{ijp\alpha} h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk}^\alpha + \sum_{ij\alpha\beta} h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}^\alpha \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta}H^\beta - 2n^2H^4 \\
& - 2 \sum_{\alpha\beta} (N(\hat{A}_\alpha\hat{A}_\beta - \hat{A}_\beta\hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta).
\end{aligned}$$

- 在一般流形之中且  $p=1$  时

$$\begin{aligned}
S_{,kl} &= \sum_{ij} 2h_{ij} \bar{R}_{(n+1)ijk, l} - \sum_{ij} 2h_{ij} \bar{R}_{(n+1)kli, j} + \sum_{ij} 2h_{ij} h_{kl, ij} + \sum_{ij} 2h_{ij, k} h_{ij, l} \\
&+ 2 \left[ \sum_{ijp} h_{ij} h_{pk} \bar{R}_{ijpl} + \sum_{ijp} h_{ij} h_{ip} \bar{R}_{kpjl} - S \sum_p h_{kp} h_{pl} \right. \\
&\left. + \sum_{ijp} (h_{ij} h_{il} h_{kp} h_{pj} + h_{ij} h_{ip} h_{pj} h_{kl} - h_{ij} h_{ip} h_{pl} h_{jk}) \right], \\
\Delta S &= \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)ijk, k} - \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)kka, j} + \sum_{ij} 2nh_{ij} H_{,ij} + 2 |Dh|^2 \\
&+ \sum_{ijkl} 2h_{ij} h_{kl} \bar{R}_{ijkl} + \sum_{ijkl} 2h_{ij} h_{il} \bar{R}_{jkkl} - 2S^2 + 2nP_3 H.
\end{aligned}$$

- 在空间形式之中且  $p \geq 2$  时

$$\begin{aligned}
S_{,kl} &= \sum_{ij\alpha} 2h_{ij}^\alpha h_{kl, ij}^\alpha + \sum_{ij\alpha} 2h_{ij, k}^\alpha h_{ij, l}^\alpha \\
&+ 2 \left[ \sum_\alpha -cnH^\alpha h_{kl}^\alpha + c\delta_{kl} S \right. \\
&+ \sum_{ijp\alpha\beta} (h_{ij}^\alpha h_{il}^\beta h_{kp}^\alpha h_{pj}^\beta - h_{ij}^\alpha h_{ij}^\beta h_{kp}^\alpha h_{pl}^\beta) \\
&+ \sum_{ijp\alpha\beta} (h_{ij}^\alpha h_{ip}^\alpha h_{pj}^\beta h_{kl}^\beta - h_{ij}^\alpha h_{ip}^\alpha h_{pl}^\beta h_{jk}^\beta) \\
&\left. + \sum_{ijp\alpha\beta} (h_{ij}^\alpha h_{ik}^\beta h_{jp}^\beta h_{pl}^\alpha - h_{ij}^\alpha h_{ik}^\beta h_{jp}^\alpha h_{pl}^\beta) \right], \\
\Delta S &= \sum_{ij\alpha} 2nh_{ij}^\alpha H_{,ij}^\alpha + 2 |Dh|^2 + 2ncS - 2n^2 cH^2 + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta}H^\beta \\
&- 2 \sum_{\alpha\beta} (N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) \\
&= \sum_{ij\alpha} 2nh_{ij}^\alpha H_{,ij}^\alpha + 2 |Dh|^2 + 2ncS - 2n^2 cH^2 + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta}H^\beta - 2n^2 H^4 \\
&- 2 \sum_{\alpha\beta} (N(\hat{A}_\alpha\hat{A}_\beta - \hat{A}_\beta\hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta).
\end{aligned}$$

- 在空间形式之中且  $p=1$  时

$$\begin{aligned}
S_{,kl} &= \sum_{ij} 2h_{ij} h_{kl, ij} + \sum_{ij} 2h_{ij, k} h_{ij, l} \\
&- 2cnHh_{kl} + 2c\delta_{kl} S - 2S \sum_p h_{kp} h_{pl} \\
&+ \sum_{ijp} 2(h_{ij} h_{il} h_{kp} h_{pj} + h_{ij} h_{ip} h_{pj} h_{kl} - h_{ij} h_{ip} h_{pl} h_{jk}), \\
\Delta S &= \sum_{ij} 2nh_{ij} H_{,ij} + 2 |Dh|^2 - 2n^2 cH^2 + 2ncS - 2S^2 + 2nHP_3.
\end{aligned}$$

引理 10.6: 对于函数  $G(S)$ , 我们有计算

- 原流形为一般流形, 余维数大于等于 2 时

$$\begin{aligned}
 \Delta G(S) &= G'(S) |\nabla S|^2 + G'(S) \left( \sum_{ijk\alpha} -2h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + \sum_{ijk\alpha} 2h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha} + \sum_{ij\alpha} 2nh_{ij}^{\alpha} H_{,ij}^{\alpha} + 2|Dh|^2 \right. \\
 &\quad + 2 \left( \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + \sum_{ijk\alpha\beta} h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk} \right) \\
 &\quad \left. + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta} H^{\beta} - 2 \sum_{\alpha\beta} [N(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha}) + (S_{\alpha\beta})^2] \right) \\
 &= G'(S) |\nabla S|^2 + G'(S) \left( \sum_{ijk\alpha} -2h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} \right. \\
 &\quad + \sum_{ijk\alpha} 2h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha} + \sum_{ij\alpha} 2nh_{ij}^{\alpha} H_{,ij}^{\alpha} + 2|Dh|^2 \\
 &\quad + 2 \left( \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + \sum_{ijpk\alpha} h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + \sum_{ijk\alpha\beta} h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk} \right) \\
 &\quad + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta} H^{\beta} - 2n^2 H^4 \\
 &\quad \left. - 2 \sum_{\alpha\beta} [N(\hat{A}_{\alpha}\hat{A}_{\beta} - \hat{A}_{\beta}\hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^{\alpha} H^{\beta}] \right).
 \end{aligned}$$

- 原流形为一般流形, 余维数等于 1 时

$$\begin{aligned}
 \Delta G(S) &= G'(S) |\nabla S|^2 + G'(S) \left( \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)ijk, k} - \sum_{ijk} 2h_{ij} \bar{R}_{(n+1)kki, j} \right. \\
 &\quad + \sum_{ij} 2nh_{ij} H_{,ij} + 2|Dh|^2 + \sum_{ijkl} 2h_{ij} h_{kl} \bar{R}_{iljk} + \sum_{ijkl} 2h_{ij} h_{il} \bar{R}_{jkl} \\
 &\quad \left. - 2S^2 + 2nP_3 H \right).
 \end{aligned}$$

- 原流形为空间形式, 余维数大于等于 2 时

$$\begin{aligned}
 \Delta G(S) &= G'(S) |\nabla S|^2 + G'(S) \left\{ \sum_{ij\alpha} 2nh_{ij}^{\alpha} H_{,ij}^{\alpha} + 2|Dh|^2 + 2ncS - 2n^2 cH^2 \right. \\
 &\quad \left. + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta} H^{\beta} - 2 \sum_{\alpha\beta} [N(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha}) + (S_{\alpha\beta})^2] \right\} \\
 &= G'(S) |\nabla S|^2 + G'(S) \left\{ \sum_{ij\alpha} 2nh_{ij}^{\alpha} H_{,ij}^{\alpha} + 2|Dh|^2 + 2ncS - 2n^2 cH^2 \right. \\
 &\quad + \sum_{\alpha\beta} 2nS_{\alpha\alpha\beta} H^{\beta} - 2n^2 H^4 \\
 &\quad \left. - 2 \sum_{\alpha\beta} [N(\hat{A}_{\alpha}\hat{A}_{\beta} - \hat{A}_{\beta}\hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^{\alpha} H^{\beta}] \right\}.
 \end{aligned}$$

- 原流形为空间形式, 余维数等于 1 时

$$\begin{aligned}
 \Delta G(S) &= G'(S) |\nabla S|^2 + G'(S) \left( \sum_{ij} 2nh_{ij} H_{,ij} \right. \\
 &\quad \left. + 2|Dh|^2 - 2n^2 cH^2 + 2ncS - 2S^2 + 2nHP_3 \right).
 \end{aligned}$$

结合  $GD_{(n, F)}$  子流形的一阶变分公式, 我们可以耦合上面的计算为以下引理。

**引理 10.7:** 对于函数  $G(S)$ , 我们耦合  $GD_{(n, F)}$  - 子流形的一阶变分公式有:

- 原流形为一般流形, 余维数大于等于 2 时

$$\begin{aligned}
\Delta G(S) &= G'(S) |\nabla S|^2 + [(2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha - 2nF'(S)h_{ij}^\beta\bar{R}_{ij\alpha}^\beta H^\alpha \\
&\quad - n^2F(S)H^2] + 2n(G'(S) - F'(S))h_{ij}^\alpha H_{,ij}^\alpha + 2n(G'(S) - F'(S))S_{\alpha\beta\beta}H^\alpha \\
&\quad + 2G'(S)|Dh|^2 + n^2F(S)H^2 + 2nF'(S)h_{ij}^\beta\bar{R}_{ij\alpha}^\beta H^\alpha \\
&\quad + 2G'(S)(-h_{ij}^\alpha\bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha\bar{R}_{kki, j}^\alpha) \\
&\quad + 2G'(S)(h_{ij}^\alpha h_{pk}^\alpha\bar{R}_{ijpk} + h_{ij}^\alpha h_{ip}^\alpha\bar{R}_{kpyk} + h_{ij}^\alpha h_{ik}^\beta\bar{R}_{\alpha\beta k}^\beta) \\
&\quad - 2G'(S)(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) \\
&= G'(S) |\nabla S|^2 + [(2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\
&\quad - 2nF'(S)h_{ij}^\beta\bar{R}_{ij\alpha}^\beta H^\alpha - n^2F(S)H^2] \\
&\quad + 2n(G'(S) - F'(S))h_{ij}^\alpha H_{,ij}^\alpha + 2n(G'(S) - F'(S))S_{\alpha\beta\beta}H^\alpha \\
&\quad + 2G'(S)|Dh|^2 - 2n^2G'(S)H^4 + n^2F(S)H^2 + 2nF'(S)h_{ij}^\beta\bar{R}_{ij\alpha}^\beta H^\alpha \\
&\quad + 2G'(S)(-h_{ij}^\alpha\bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha\bar{R}_{kki, j}^\alpha) \\
&\quad + 2G'(S)(h_{ij}^\alpha h_{pk}^\alpha\bar{R}_{ijpk} + h_{ij}^\alpha h_{ip}^\alpha\bar{R}_{kpyk} + h_{ij}^\alpha h_{ik}^\beta\bar{R}_{\alpha\beta k}^\beta) \\
&\quad - 2G'(S)[N(\hat{A}_\alpha\hat{A}_\beta - \hat{A}_\beta\hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta].
\end{aligned}$$

- 原流形为一般流形, 余维数等于 1 时

$$\begin{aligned}
\Delta G(S) &= G'(S) |\nabla S|^2 + [(2nF'(S)h_{ij})H_{,ij} + 2nF'(S)P_3H \\
&\quad + 2nF'(S)h_{ij}\bar{R}_{i(n+1)(n+1)j}H - n^2F(S)H^2] \\
&\quad + 2n(G'(S) - F'(S))h_{ij}H_{,ij} + 2n(G'(S) - F'(S))P_3H \\
&\quad + 2G'(S)|Dh|^2 - 2G'(S)S^2 - 2nF'(S)h_{ij}\bar{R}_{i(n+1)(n+1)j}H + n^2F(S)H^2 \\
&\quad + 2G'(S)(h_{ij}\bar{R}_{(n+1)ijk, k} - h_{ij}\bar{R}_{(n+1)kki, j}) + 2G'(S)(h_{ij}h_{kl}\bar{R}_{ijkl} + h_{ij}h_{kl}\bar{R}_{jklk})
\end{aligned}$$

- 原流形为空间形式, 余维数大于等于 2 时

$$\begin{aligned}
\Delta G(S) &= G'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha \\
&\quad + 2nF'(S)S_{\alpha\beta\beta}H^\alpha + 2n^2cF'(S)H^2 - n^2F(S)H^2) \\
&\quad + 2n(G'(S) - F'(S))h_{ij}^\alpha H_{,ij}^\alpha - 2n^2c(G'(S) + F'(S))H^2 \\
&\quad + 2n(G'(S) - F'(S))S_{\alpha\beta\beta}H^\alpha \\
&\quad + 2G'(S)|Dh|^2 + 2ncG'(S)S + n^2F(S)H^2 \\
&\quad - 2G'(S)(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) \\
&= G'(S) |\nabla S|^2 + [(2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha \\
&\quad + 2nF'(S)S_{\alpha\beta\beta}H^\alpha + 2n^2cF'(S)H^2 - n^2F(S)H^2] \\
&\quad + 2n(G'(S) - F'(S))h_{ij}^\alpha H_{,ij}^\alpha - 2n^2c(G'(S) + F'(S))H^2 \\
&\quad + 2n(G'(S) - F'(S))S_{\alpha\beta\beta}H^\alpha \\
&\quad + 2G'(S)|Dh|^2 + 2ncG'(S)S + n^2F(S)H^2 - 2n^2G'(S)H^4
\end{aligned}$$



$$-2G'(S)(N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta).$$

- 原流形为空间形式, 余维数等于 1 时

$$\begin{aligned}\Delta G(S) &= G'(S)|\nabla S|^2 + ((2nF'(S)h_{ij})H_{,ij} + 2nF'(S)P_3H \\ &\quad + 2n^2cF'(S)H^2 - n^2F(S)H^2) + 2n(G'(S) - F'(S))h_{ij}H_{,ij} \\ &\quad + (G'(S) - F'(S))P_3H - 2n^2c(G'(S) + F'(S))H^2 \\ &\quad + 2G'(S)|Dh|^2 + 2ncG'(S)S - 2G'(S)S^2 + n^2F(S)H^2.\end{aligned}$$

特别地, 如果函数  $G(S) = F(S)$ , 则上面的耦合计算可以变为

**引理 10.8:** 对于函数  $F(S)$ , 我们耦合  $GD_{(n, F)}$ -子流形的一阶变分公式有

- 原流形为一般流形, 余维数大于等于 2 时

$$\begin{aligned}\Delta F(S) &= F'(S)|\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ &\quad - 2nF'(S)h_{ij}^\beta\bar{R}_{i\alpha}^\beta H^\alpha - n^2F(S)H^2) \\ &\quad + 2F'(S)|Dh|^2 + n^2F(S)H^2 + 2nF'(S)h_{ij}^\beta\bar{R}_{i\alpha}^\beta H^\alpha \\ &\quad + 2F'(S)(-h_{ij}^\alpha\bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha\bar{R}_{kki, j}^\alpha) \\ &\quad + 2F'(S)(h_{ij}^\alpha h_{pk}^\alpha\bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha\bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta\bar{R}_{\alpha\beta jk}) \\ &\quad - 2F'(S)(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) \\ &= F'(S)|\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ &\quad - 2nF'(S)h_{ij}^\beta\bar{R}_{i\alpha}^\beta H^\alpha - n^2F(S)H^2) \\ &\quad + 2F'(S)|Dh|^2 - 2n^2F'(S)H^4 + n^2F(S)H^2 + 2nF'(S)h_{ij}^\beta\bar{R}_{i\alpha}^\beta H^\alpha \\ &\quad + 2F'(S)(-h_{ij}^\alpha\bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha\bar{R}_{kki, j}^\alpha) \\ &\quad + 2F'(S)(h_{ij}^\alpha h_{pk}^\alpha\bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha\bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta\bar{R}_{\alpha\beta jk}) \\ &\quad - 2F'(S)(N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta).\end{aligned}$$

- 原流形为一般流形, 余维数等于 1 时

$$\begin{aligned}\Delta F(S) &= F'(S)|\nabla S|^2 + ((2nF'(S)h_{ij})H_{,ij} + 2nF'(S)P_3H \\ &\quad + 2nF'(S)h_{ij}\bar{R}_{i(n+1)(n+1)j}H - n^2F(S)H^2) \\ &\quad + 2F'(S)|Dh|^2 - 2F'(S)S^2 - 2nF'(S)h_{ij}\bar{R}_{i(n+1)(n+1)j}H + n^2F(S)H^2 \\ &\quad + 2F'(S)(h_{ij}\bar{R}_{(n+1)ijk, k} - h_{ij}\bar{R}_{(n+1)kki, j}) + 2F'(S)(h_{ij}h_{kl}\bar{R}_{ijkl} + h_{ij}h_{il}\bar{R}_{jjkl})\end{aligned}$$

- 原流形为空间形式, 余维数大于等于 2 时

$$\begin{aligned}\Delta F(S) &= F'(S)|\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha \\ &\quad + 2nF'(S)S_{\alpha\beta\beta}H^\alpha + 2n^2cF'(S)H^2 - n^2F(S)H^2) \\ &\quad - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2F(S)H^2 \\ &\quad - 2F'(S)(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2)\end{aligned}$$

$$\begin{aligned}
&= F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha \\
&\quad + 2nF'(S)S_{\alpha\beta}H^\alpha + 2n^2cF'(S)H^2 - n^2F(S)H^2) \\
&\quad - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2F(S)H^2 - 2n^2F'(S)H^4 \\
&\quad - 2F'(S)(N(\hat{A}_\alpha\hat{A}_\beta - \hat{A}_\beta\hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta).
\end{aligned}$$

- 原流形为空间形式, 余维数等于 1 时

$$\begin{aligned}
\Delta F(S) &= F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij})H_{,ij} + 2nF'(S)P_3H \\
&\quad + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 \\
&\quad + 2ncF'(S)S - 2F'(S)S^2 + n^2F(S)H^2.
\end{aligned}$$

**引理 10.9:** 假设符号如上文所述, 对于上面的引理之中出现的某些项, 我们有如下两个估计。

- 陈省身类型估计 I (余维数大于等于 2)

$$N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2 \leq (2 - \frac{1}{p})S^2.$$

等号成立当且仅当下面任意一种情形成立

**情形 1:** 所有的矩阵  $A_\alpha = 0, \forall \alpha$ .

**情形 2:** 余维数为 2, 并且矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 陈省身类型估计 II (余维数大于等于 2)

$$2n\hat{S}_{\alpha\beta}H^\beta H^\alpha + N(\hat{A}_\alpha\hat{A}_\beta - \hat{A}_\beta\hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 \leq 2n\rho H^2 + (2 - \frac{1}{p})\rho^2.$$

等号成立当且仅当下面任意一种情形成立

**情形 1:** 所有的矩阵  $\hat{A}_\alpha = 0, \forall \alpha$ .

**情形 2:** 余维数为 2, 并且矩阵  $\hat{A}_{n+1} \neq 0, \hat{A}_{n+2} \neq 0, \hat{S}_{(n+1)(n+1)} = \hat{S}_{(n+2)(n+2)} = \frac{\rho}{2}, \vec{H} = 0$ , 并且

$$\hat{A}_{n+1} = \hat{A}_{n+1} = \frac{\sqrt{\rho}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \hat{A}_{n+2} = \hat{A}_{n+2} = \frac{\sqrt{\rho}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 李安民类型估计 I (余维数大于等于 2)

$$N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2 \leq \frac{3}{2} S^2.$$

等号成立当且仅当下面任意一种情形成立

情形 1: 所有的矩阵  $A_\alpha = 0, \forall \alpha$ .

情形 2: 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S}{2}, A_{n+3} = A_{n+4} = \dots$

$= A_{n+p} = 0, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 李安民类型估计 II (余维数大于等于 2)

$$2n\hat{S}_{\alpha\beta}H^\alpha H^\beta + N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 \leq 2n\rho H^2 + \frac{3}{2}\rho^2.$$

等号成立当且仅当下面任意一种情形成立

情形 1: 所有的矩阵  $\hat{A}_\alpha = 0, \forall \alpha$ .

情形 2: 矩阵  $\hat{A}_{n+1} \neq 0, \hat{A}_{n+2} \neq 0, \hat{S}_{(n+1)(n+1)} = \hat{S}_{(n+2)(n+2)} = \frac{\rho}{2}, \hat{A}_{n+3} = \hat{A}_{n+4} = \dots$

$= \hat{A}_{n+p} = 0, \vec{H} = 0$ , 并且

$$\hat{A}_{n+1} = \hat{A}_{n+1} = \frac{\sqrt{\rho}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \hat{A}_{n+2} = \hat{A}_{n+2} = \frac{\sqrt{\rho}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

证明: 首先我们证明陈省身估计。我们用符号  $T_1$  来表示需要估计的项

$$T_1 = 2n \sum_{\alpha\beta} \hat{S}_{\alpha\beta} H^\alpha H^\beta + \sum_{\alpha\beta} (\hat{S}_{\alpha\beta})^2 + \sum_{\alpha \neq \beta} 2\hat{S}_{\alpha\alpha} \hat{S}_{\beta\beta}.$$

对角化  $(\hat{S}_{\alpha\beta})$  使得  $\hat{S}_{\alpha\beta} = 0, \alpha \neq \beta$ , 那么我们有

$$T_1 = 2nT_2 + T_3, (*)$$

此处

$$T_2 = \sum_{\alpha} (\hat{S}_{\alpha\alpha}) (H^\alpha)^2$$

和

$$T_3 = \sum_{\alpha} (\hat{S}_{\alpha\alpha})^2 + \sum_{\alpha \neq \beta} 2\hat{S}_{\alpha\alpha}\hat{S}_{\beta\beta}.$$

我们推得

$$T_2 = \sum_{\alpha} (\hat{S}_{\alpha\alpha}) (H^{\alpha})^2 \leq \sum_{\alpha} (\hat{S}_{\alpha\alpha}) \sum_{\beta} (H^{\beta})^2 \leq \rho H^2$$

和

$$\begin{aligned} T_3 &= \sum_{\alpha} (\hat{S}_{\alpha\alpha})^2 + \sum_{\alpha \neq \beta} 2\hat{S}_{\alpha\alpha}\hat{S}_{\beta\beta} \\ &= \sum_{\alpha} (\hat{S}_{\alpha\alpha})^2 + \sum_{\alpha \neq \beta} \hat{S}_{\alpha\alpha}\hat{S}_{\beta\beta} + \sum_{\alpha \neq \beta} \hat{S}_{\alpha\alpha}\hat{S}_{\beta\beta} \\ &= \left( \sum_{\alpha} \hat{S}_{\alpha\alpha} \right)^2 + \sum_{\alpha=1}^p \hat{S}_{\alpha\alpha} \left( \sum_{\beta} \hat{S}_{\beta\beta} - \hat{S}_{\alpha\alpha} \right) \\ &= \hat{S}^2 + \hat{S}^2 - \frac{1}{p} \cdot p \cdot \sum_{\alpha=1}^p \hat{S}_{\alpha\alpha}^2 \\ &\leq 2\hat{S}^2 - \frac{1}{p} \left( \left( \sum_{\alpha} \hat{S}_{\alpha\alpha} \right)^2 \right) = \left( 2 - \frac{1}{p} \right) \hat{S}^2 \\ &= \left( 2 - \frac{1}{p} \right) \rho^2. \end{aligned}$$

代  $T_2$  和  $T_3$  进入 (\*), 可得

$$T_1 \leq 2n\rho H^2 + \left( 2 - \frac{1}{p} \right) \rho^2.$$

如果等号成立, 那么意味着上面推导过程之中的所有不等号都成立, 即是

$$\begin{aligned} \sum_{\alpha \neq \beta} \hat{S}_{\alpha\alpha} (H^{\beta})^2 &= 0, \\ N(\hat{A}_{\alpha}\hat{A}_{\beta} - \hat{A}_{\beta}\hat{A}_{\alpha}) &= 2N(\hat{A}_{\alpha})N(\hat{A}_{\beta}), \\ \hat{S}_{\alpha\alpha} &= \hat{S}_{\beta\beta} = \frac{\rho}{p}, \quad \forall \alpha \neq \beta. \end{aligned}$$

如果某个矩阵  $\hat{A}_{\alpha} = 0$ , 那么其他所有的矩阵都为 0, 此时不等式变为等式。如果某个矩阵  $\hat{A}_{\alpha} \neq 0$ , 那么所有的矩阵都不为 0, 此时根据陈省身不等式的后面的结论, 如果余维数大于 2, 那么某个矩阵  $\hat{A}_{\alpha}$  必须为 0, 与假设矛盾, 因此余维数一定为 2, 并且矩阵可以同时对角化为上面引理之中的情形, 在此种情形, 平均曲率向量为 0。由此, 陈省身估计得证。对于李安民估计可用李安民不等式同理可证。

**引理 10.10:** 结合上面的两个引理, 我们有关于函数  $F(S)$  的二阶导数  $\Delta F(S)$  的估计。

• 原流形为一般流形, 余维数等于 1 时

$$\begin{aligned} \Delta F(S) &= F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij})H_{,ij} + 2nF'(S)P_3H \\ &\quad + 2nF'(S)h_{ij}\bar{R}_{i(n+1)(n+1)j}H - n^2F(S)H^2) \end{aligned}$$

$$+ 2F'(S) |Dh|^2 - 2F'(S) S^2 - 2nF'(S) h_{ij} \bar{R}_{i(n+1)(n+1)j} H + n^2 F(S) H^2 \\ + 2F'(S) (h_{ij} \bar{R}_{(n+1)ijk, k} - h_{ij} \bar{R}_{(n+1)kki, j}) + 2F'(S) (h_{ij} h_{kl} \bar{R}_{ijkl} + h_{ij} h_{il} \bar{R}_{jkil})$$

- 原流形为一般流形, 余维数大于等于 2 时

$$\Delta F(S) = F'(S) |\nabla S|^2 + ((2nF'(S) h_{ij}^\alpha) H_{,ij}^\alpha + 2nF'(S) S_{\alpha\beta\beta} H^\alpha \\ - 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha - n^2 F(S) H^2) + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\ + 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ - 2F'(S) (N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) \\ = F'(S) |\nabla S|^2 + ((2nF'(S) h_{ij}^\alpha) H_{,ij}^\alpha + 2nF'(S) S_{\alpha\beta\beta} H^\alpha \\ - 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha - n^2 F(S) H^2) \\ + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 + n^2 F(S) H^2 + 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha \\ + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ - 2F'(S) (N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^\alpha H^\beta).$$

- 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计

I 为

$$\Delta F(S) \geq F'(S) |\nabla S|^2 + ((2nF'(S) h_{ij}^\alpha) H_{,ij}^\alpha + 2nF'(S) S_{\alpha\beta\beta} H^\alpha \\ - 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha - n^2 F(S) H^2) + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\ + 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) - 2F'(S) (2 - \frac{1}{p}) S^2.$$

当  $S = S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

0,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计

II 为

$$\Delta F(S) \geq F'(S) |\nabla S|^2 + ((2nF'(S) h_{ij}^\alpha) H_{,ij}^\alpha + 2nF'(S) S_{\alpha\beta\beta} H^\alpha$$

$$\begin{aligned}
& -2nF'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} - n^2F(S)H^2) \\
& + 2F'(S)|Dh|^2 - 2n^2F'(S)H^4 + n^2F(S)H^2 + 2nF'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} \\
& + 2F'(S)(-h_{ij}^{\alpha}\bar{R}_{ijk,k}^{\alpha} + h_{ij}^{\alpha}\bar{R}_{kkl,j}^{\alpha}) \\
& + 2F'(S)(h_{ij}^{\alpha}h_{pk}^{\alpha}\bar{R}_{ipjk} + h_{ij}^{\alpha}h_{ip}^{\alpha}\bar{R}_{kpjk} + h_{ij}^{\alpha}h_{ik}^{\beta}\bar{R}_{\alpha\beta jk}) \\
& - 2F'(S)(2n\rho H^2 + (2 - \frac{1}{p})\rho^2).
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计

I 为

$$\begin{aligned}
\Delta F(S) & \leq F'(S)|\nabla S|^2 + (2nF'(S)h_{ij}^{\alpha})H_{,ij}^{\alpha} + 2nF'(S)S_{\alpha\beta\beta}H^{\alpha} \\
& - 2nF'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} - n^2F(S)H^2 + 2F'(S)|Dh|^2 + n^2F(S)H^2 \\
& + 2nF'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} + 2F'(S)(-h_{ij}^{\alpha}\bar{R}_{ijk,k}^{\alpha} + h_{ij}^{\alpha}\bar{R}_{kkl,j}^{\alpha}) \\
& + 2F'(S)(h_{ij}^{\alpha}h_{pk}^{\alpha}\bar{R}_{ipjk} + h_{ij}^{\alpha}h_{ip}^{\alpha}\bar{R}_{kpjk} + h_{ij}^{\alpha}h_{ik}^{\beta}\bar{R}_{\alpha\beta jk}) - 2F'(S)(2 - \frac{1}{p})S^2.
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计

II 为

$$\begin{aligned}
\Delta F(S) & \leq F'(S)|\nabla S|^2 + (2nF'(S)h_{ij}^{\alpha})H_{,ij}^{\alpha} + 2nF'(S)S_{\alpha\beta\beta}H^{\alpha} \\
& - 2nF'(S)h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} - n^2F(S)H^2 + 2F'(S)|Dh|^2 - 2n^2F'(S)H^4
\end{aligned}$$

$$\begin{aligned}
& + n^2 F(S) H^2 + 2nF'(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} + 2F'(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + 2F'(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\
& - 2F'(S) (2n\rho H^2 + (2 - \frac{1}{p})\rho^2).
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计 I 为

$$\begin{aligned}
\Delta F(S) & \geq F'(S) |\nabla S|^2 + (2nF'(S) h_{ij}^{\alpha}) H_{,ij}^{\alpha} + 2nF'(S) S_{\alpha\beta\beta} H^{\alpha} \\
& - 2nF'(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} - n^2 F(S) H^2 + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\
& + 2nF'(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} + 2F'(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + 2F'(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) - 3F'(S) S^2
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计 II 为

$$\begin{aligned}
\Delta F(S) & \geq F'(S) |\nabla S|^2 + (2nF'(S) h_{ij}^{\alpha}) H_{,ij}^{\alpha} + 2nF'(S) S_{\alpha\beta\beta} H^{\alpha} \\
& - 2nF'(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} - n^2 F(S) H^2 + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 \\
& + n^2 F(S) H^2 + 2nF'(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} + 2F'(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + 2F'(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) - 2F'(S) (2n\rho H^2 + \frac{3}{2}\rho^2).
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于2时,  $F'(u) \leq 0$  时, 李安民类型估计

I 为

$$\begin{aligned} \Delta F(S) \leq & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ & - 2nF'(S)h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha - n^2 F(S)H^2) + 2F'(S)|Dh|^2 + n^2 F(S)H^2 \\ & + 2nF'(S)h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S)(-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S)(h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta k}) - 3F'(S)S^2 \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于2时,  $F'(u) \leq 0$  时, 李安民类型估计

II 为

$$\begin{aligned} \Delta F(S) \leq & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ & - 2nF'(S)h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha - n^2 F(S)H^2) + 2F'(S)|Dh|^2 - 2n^2 F'(S)H^4 \\ & + n^2 F(S)H^2 + 2nF'(S)h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S)(-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S)(h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta k}) - 2F'(S)(2npH^2 + \frac{3}{2}p^2). \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且



$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数等于 1 时

$$\begin{aligned} \Delta F(S) = & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij})H_{,ij} + 2nF'(S)P_3H \\ & + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 \\ & + 2ncF'(S)S - 2F'(S)S^2 + n^2F(S)H^2. \end{aligned}$$

- 原流形为空间形式, 余维数大于等于 2 时

$$\begin{aligned} \Delta F(S) = & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha \\ & + 2nF'(S)S_{\alpha\beta\beta}H^\alpha + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 \\ & + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2F(S)H^2 \\ & - 2F'(S)(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) \\ = & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha \\ & + 2nF'(S)S_{\alpha\beta\beta}H^\alpha + 2n^2cF'(S)H^2 - n^2F(S)H^2) \\ & - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2F(S)H^2 - 2n^2F'(S)H^4 \\ & - 2F'(S)(N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2\hat{n}\hat{S}_{\alpha\beta}H^\alpha H^\beta). \end{aligned}$$

- 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计

I 为

$$\begin{aligned} \Delta F(S) \geq & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ & + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 \\ & + 2ncF'(S)S + n^2F(S)H^2 - 2(2 - \frac{1}{p})F'(S)S^2. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计

II 为

$$\Delta F(S) \geq F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha$$

$$\begin{aligned}
& + 2n^2 cF'(S)H^2 - n^2 F(S)H^2 \\
& - 4n^2 cF'(S)H^2 + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2 F(S)H^2 \\
& - 2n^2 F'(S)H^4 - 2F'(S)(2npH^2 + (2 - \frac{1}{p})\rho^2).
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计

I 为

$$\begin{aligned}
\Delta F(S) \leq & F'(S)|\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\
& + 2n^2 cF'(S)H^2 - n^2 F(S)H^2) - 4n^2 cF'(S)H^2 + 2F'(S)|Dh|^2 \\
& + 2ncF'(S)S + n^2 F(S)H^2 - 2(2 - \frac{1}{p})F'(S)S^2.
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计

II 为

$$\begin{aligned}
\Delta F(S) \leq & F'(S)|\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\
& + 2n^2 cF'(S)H^2 - n^2 F(S)H^2) \\
& - 4n^2 cF'(S)H^2 + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2 F(S)H^2 \\
& - 2n^2 F'(S)H^4 - 2F'(S)(2npH^2 + (2 - \frac{1}{p})\rho^2).
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} \Delta F(S) &= F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ &\quad + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 \\ &\quad + 2ncF'(S)S + n^2F(S)H^2 - 3F'(S)S^2 \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} \Delta F(S) &\leq F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ &\quad + 2n^2cF'(S)H^2 - n^2F(S)H^2) \\ &\quad - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 + 2ncF'(S)S + n^2F(S)H^2 \\ &\quad - 2n^2F'(S)H^4 - 2F'(S)(2n\rho H^2 + \frac{3}{2}\rho^2). \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 李安民类型估计 I 为

$$\begin{aligned}\Delta F(S) \leq & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ & + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 \\ & + 2ncF'(S)S + n^2F(S)H^2 - 3F'(S)S^2.\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 李安民类型估计 II 为

$$\begin{aligned}\Delta F(S) \leq & F'(S) |\nabla S|^2 + ((2nF'(S)h_{ij}^\alpha)H_{,ij}^\alpha + 2nF'(S)S_{\alpha\beta\beta}H^\alpha \\ & + 2n^2cF'(S)H^2 - n^2F(S)H^2) - 4n^2cF'(S)H^2 + 2F'(S)|Dh|^2 \\ & + 2ncF'(S)S + n^2F(S)H^2 - 2n^2F'(S)H^4 - 2F'(S)(2npH^2 + \frac{3}{2}p^2).\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

## 10.2 $GD_{(n,F)}$ 子流形的 Simons 型积分不等式

对于上面引理之中的表达式我们在流形之上进行积分运算, 利用分部积分公式和  $GD_{(n,F)}$  子流形的 Euler - Lagrange 方程, 我们可得一系列的定理。

**定理 10.1:** 假设  $x: M^n \rightarrow N^{n+1}$  为一般流形之中的  $GD_{(n,F)}$  超曲面, 我们有

$$\begin{aligned}0 = & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 - 2F'(S)S^2 \\ & - 2nF'(S)h_{ij}\bar{R}_{i(n+1)(n+1)j}H + n^2F(S)H^2 \\ & + 2F'(S)(h_{ij}\bar{R}_{(n+1)ijk,k} - h_{ij}\bar{R}_{(n+1)kki,j}) + 2F'(S)(h_{ij}h_{kl}\bar{R}_{iljk} + h_{ij}h_{il}\bar{R}_{jkkl}) dv.\end{aligned}$$

**定理 10.2:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n, F)}$  子流形, 我们有

$$\begin{aligned}
 0 &= \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\
 &\quad + 2nF'(S) h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
 &\quad + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\
 &\quad - 2F'(S) (N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) dv \\
 &= \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 + n^2 F(S) H^2 \\
 &\quad + 2nF'(S) h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
 &\quad + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\
 &\quad - 2F'(S) (N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^\alpha H^\beta) dv.
 \end{aligned}$$

**定理 10.3:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n, F)}$  子流形, 我们有

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计 I 为

$$\begin{aligned}
 0 &\geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\
 &\quad + 2nF'(S) h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
 &\quad + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) - 2F'(S) (2 - \frac{1}{p}) S^2 dv.
 \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

0,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计 II 为

$$\begin{aligned}
 0 &\geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 + n^2 F(S) H^2 \\
 &\quad + 2nF'(S) h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
 &\quad + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk})
 \end{aligned}$$

$$-2F'(S)(2npH^2 + (2 - \frac{1}{p})\rho^2) dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计 I 为

$$\begin{aligned} 0 \leq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\ & + 2nF'(S) h_{ij}^\beta \bar{R}_{ija}^\beta H^a + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2F'(S) (2 - \frac{1}{p}) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 + n^2 F(S) H^2 \\ & + 2nF'(S) h_{ij}^\beta \bar{R}_{ija}^\beta H^a + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2F'(S) (2npH^2 + (2 - \frac{1}{p})\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \geq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\ & + 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 3F'(S) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \geq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 + n^2 F(S) H^2 \\ & + 2nF'(S) h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2F'(S) (2npH^2 + \frac{3}{2}p^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \leq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + n^2 F(S) H^2 \\ & + 2nF'(S) h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 3F'(S) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 - 2n^2 F'(S) H^4 + n^2 F(S) H^2 \\ & + 2nF'(S) h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2F'(S) (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2F'(S) (2n\rho H^2 + \frac{3}{2}\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$



**定理 10.4:** 假设  $x: M^n \rightarrow R^{n+1}$  为空间形式之中的  $GD_{(n, F)}$  超曲面, 我们有

$$0 = \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 + 2nc F'(S) S - 2F'(S) S^2 + n^2 F(S) H^2 dv.$$

**定理 10.5:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n, F)}$  子流形, 我们有

$$\begin{aligned} 0 &= \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 + 2nc F'(S) S + n^2 F(S) H^2 \\ &\quad - 2F'(S) (N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) dv \\ &= \int_M F'(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ &\quad + 2nc F'(S) S + n^2 F(S) H^2 - 2n^2 F'(S) H^4 \\ &\quad - 2F'(S) (N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n \hat{S}_{\alpha\beta} H^\alpha H^\beta) dv. \end{aligned}$$

**定理 10.6:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n, F)}$  子流形, 我们有

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计 I 为

$$0 \geq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 + 2nc F'(S) S + n^2 F(S) H^2 - 2(2 - \frac{1}{p}) F'(S) S^2 dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 陈省身类型估计 II 为

$$0 \geq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 + 2nc F'(S) S + n^2 F(S) H^2 - 2n^2 F'(S) H^4 - 2F'(S) (2n\rho H^2 + (2 - \frac{1}{p})\rho^2) dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计 I 为

$$0 \leq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ + 2nc F'(S) S + n^2 F(S) H^2 - 2\left(2 - \frac{1}{p}\right) F'(S) S^2 dv.$$

当  $S = S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 陈省身类型估计 II 为

$$0 \leq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ + 2nc F'(S) S + n^2 F(S) H^2 - 2n^2 F'(S) H^4 - 2F'(S) (2npH^2 + \left(2 - \frac{1}{p}\right)p^2) dv.$$

当  $S = S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计

I 为

$$0 = \int_M F'(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ + 2nc F'(S) S + n^2 F(S) H^2 - 3F'(S) S^2 \, dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \geq 0$  时, 李安民类型估计

II 为

$$0 \leq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ + 2nc F'(S) S + n^2 F(S) H^2 - 2n^2 F'(S) H^4 - 2F'(S) (2n\rho H^2 + \frac{3}{2}\rho^2) \, dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 李安民类型估计

I 为

$$0 \leq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ + 2nc F'(S) S + n^2 F(S) H^2 - 3F'(S) S^2 \, dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时,  $F'(u) \leq 0$  时, 李安民类型估计 II 为

$$0 \leq \int_M F''(S) |\nabla S|^2 - 4n^2 c F'(S) H^2 + 2F'(S) |Dh|^2 \\ + 2nc F'(S) S + n^2 F(S) H^2 - 2n^2 F'(S) H^4 - 2F'(S) (2n\rho H^2 + \frac{3}{2}\rho^2) dv.$$

当  $S=S_0>0$  且  $F'(S_0)<0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

### 10.3 $GD_{(n,r)}$ 子流形的 Simons 型积分不等式

对于  $GD_{(n,r)}$  子流形, 利用  $GD_{(n,r)}$  子流形的 Simons 不等式估计可以得到下面的定理。

**定理 10.7:** 假设  $x: M^n \rightarrow N^{n+1}$  为一般流形之中的  $GD_{(n,r)}$  超曲面且  $(M, r) \in T_{1,1} \cup T_{2,2}$ , 我们有

$$0 = \int_M r(r-1) S^{r-2} |\nabla S|^2 + 2r S^{r-1} |Dh|^2 - 2r S^{r-1} S^2 \\ - 2nr S^{r-1} h_{ij} \bar{R}_{i(n+1)(n+1)j} H + n^2 S^r H^2 \\ + 2r S^{r-1} (h_{ij} \bar{R}_{(n+1)ijk, k} - h_{ij} \bar{R}_{(n+1)kki, j}) + 2r S^{r-1} (h_{ij} h_{kl} \bar{R}_{iljk} + h_{ij} h_{il} \bar{R}_{jkl}) dv.$$

**定理 10.8:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1} \cup T_{2,2}$ , 我们有

$$0 = \int_M r(r-1) S^{r-2} |\nabla S|^2 + 2r S^{r-1} |Dh|^2 + n^2 S^r H^2 \\ + 2nr S^{r-1} h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2r S^{r-1} (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha)$$

$$\begin{aligned}
& + 2rS^{r-1}(h_{ij}^{\alpha}h_{pk}^{\alpha}\bar{R}_{ipjk} + h_{ij}^{\alpha}h_{ip}^{\alpha}\bar{R}_{kpjk} + h_{ij}^{\alpha}h_{ik}^{\beta}\bar{R}_{\alpha\beta jk}) \\
& - 2rS^{r-1}(N(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha}) + (S_{\alpha\beta})^2) dv \\
= & \int_M Fr(r-1)S^{r-2}|\nabla S|^2 + 2rS^{r-1}|Dh|^2 - 2n^2rS^{r-1}H^4 + n^2S^rH^2 \\
& + 2nrS^{r-1}h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} + 2rS^{r-1}(-h_{ij}^{\alpha}\bar{R}_{ijk,k}^{\alpha} + h_{ij}^{\alpha}\bar{R}_{kki,j}^{\alpha}) \\
& + 2rS^{r-1}(h_{ij}^{\alpha}h_{pk}^{\alpha}\bar{R}_{ipjk} + h_{ij}^{\alpha}h_{ip}^{\alpha}\bar{R}_{kpjk} + h_{ij}^{\alpha}h_{ik}^{\beta}\bar{R}_{\alpha\beta jk}) \\
& - 2rS^{r-1}(N(\hat{A}_{\alpha}\hat{A}_{\beta} - \hat{A}_{\beta}\hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^{\alpha}H^{\beta}) dv.
\end{aligned}$$

**定理 10.9:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1} \cup T_{2,2}$ , 我们有

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 陈省身类型估计 I 为

$$\begin{aligned}
0 \geq & \int_M r(r-1)S^{r-2}|\nabla S|^2 + 2rS^{r-1}|Dh|^2 + n^2S^rH^2 \\
& + 2nrS^{r-1}h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} + 2rS^{r-1}(-h_{ij}^{\alpha}\bar{R}_{ijk,k}^{\alpha} + h_{ij}^{\alpha}\bar{R}_{kki,j}^{\alpha}) \\
& + 2rS^{r-1}(h_{ij}^{\alpha}h_{pk}^{\alpha}\bar{R}_{ipjk} + h_{ij}^{\alpha}h_{ip}^{\alpha}\bar{R}_{kpjk} + h_{ij}^{\alpha}h_{ik}^{\beta}\bar{R}_{\alpha\beta jk}) \\
& - 2rS^{r-1}(2 - \frac{1}{p})S^2 dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 陈省身类型估计 II 为

$$\begin{aligned}
0 \geq & \int_M r(r-1)S^{r-2}|\nabla S|^2 + 2rS^{r-1}|Dh|^2 - 2n^2rS^{r-1}H^4 + n^2S^rH^2 \\
& + 2nrS^{r-1}h_{ij}^{\beta}\bar{R}_{ij\alpha}^{\beta}H^{\alpha} + 2rS^{r-1}(-h_{ij}^{\alpha}\bar{R}_{ijk,k}^{\alpha} + h_{ij}^{\alpha}\bar{R}_{kki,j}^{\alpha}) \\
& + 2rS^{r-1}(h_{ij}^{\alpha}h_{pk}^{\alpha}\bar{R}_{ipjk} + h_{ij}^{\alpha}h_{ip}^{\alpha}\bar{R}_{kpjk} + h_{ij}^{\alpha}h_{ik}^{\beta}\bar{R}_{\alpha\beta jk}) \\
& - 2rS^{r-1}(2n\rho H^2 + (2 - \frac{1}{p})\rho^2) dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅

当: 余维数为 2, 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}, r \leq 0$  时, 陈省身类型估计 I 为

$$\begin{aligned} 0 \leq & \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 + n^2 S^r H^2 \\ & + 2nrS^{r-1} h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2rS^{r-1} (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2rS^{r-1} (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2rS^{r-1} (2 - \frac{1}{p}) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵

$A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}, r \leq 0$  时, 陈省身类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 - 2n^2 rS^{r-1} H^4 + n^2 S^r H^2 \\ & + 2nrS^{r-1} h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2rS^{r-1} (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2rS^{r-1} (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2rS^{r-1} (2n\rho H^2 + (2 - \frac{1}{p})\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r < 0$  时, 等式成立当且仅当: 余维数为 2, 矩

阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \geq & \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 + n^2 S^r H^2 \\ & + 2nrS^{r-1} h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2rS^{r-1} (-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2rS^{r-1} (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 3rS^{r-1} S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} =$

$0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \geq & \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 - 2n^2 rS^{r-1} H^4 + n^2 S^r H^2 \\ & + 2nrS^{r-1} h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2rS^{r-1} (-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2rS^{r-1} (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2rS^{r-1} (2n\rho H^2 + \frac{3}{2}\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} =$

$0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \leq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \leq & \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 + n^2 S^r H^2 \\ & + 2nrS^{r-1} h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2F'(S) (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2rS^{r-1} (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) - 3rS^{r-1} S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,

$A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \leq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 - 2n^2 rS^{r-1} H^4 + n^2 S^r H^2 \\ & + 2nrS^{r-1} h_{ij}^\beta \bar{R}_{ij\alpha}^\beta H^\alpha + 2rS^{r-1} (-h_{ij}^\alpha \bar{R}_{ijk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\ & + 2rS^{r-1} (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\ & - 2rS^{r-1} (2n\rho H^2 + \frac{3}{2}\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,

$A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$



**定理 10.10:** 假设  $x: M^n \rightarrow R^{n+1}$  为空间形式之中的  $GD_{(n,r)}$  超曲面且  $(M, r) \in T_{1,1} \cup T_{2,2}$ , 我们有

$$0 = \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S - 2rS^{r-1}S^2 + n^2S^rH^2 dv.$$

**定理 10.11:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1} \cup T_{2,2}$ , 我们有

$$0 = \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 + 2ncrS^{r-1}S + n^2S^rH^2 \\ - 2rS^{r-1}(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) dv \\ = \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S^rH^2 - 2n^2rS^{r-1}H^4 \\ - 2rS^{r-1}(N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta}H^\alpha H^\beta) dv.$$

**定理 10.12:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1} \cup T_{2,2}$ , 我们有

- 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 陈省身类型估计 I 为

$$0 \geq \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S^rH^2 - 2(2 - \frac{1}{p})rS^{r-1}S^2 dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 陈省身类型估计 II 为

$$0 \geq \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S^rH^2 - 2n^2rS^{r-1}H^4 - 2rS^{r-1}(2npH^2 + (2 - \frac{1}{p})\rho^2) dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当:

余维数为 2, 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}, r \leq 0$  时, 陈省身类型估计 I 为

$$0 \leq \int_M r(r-1)S^{r-2} |\nabla S|^2 - 4n^2 crS^{r-1}H^2 + 2rS^{r-1} |Dh|^2 \\ + 2ncrS^{r-1}S + n^2 S^r H^2 - 2\left(2 - \frac{1}{p}\right)rS^{r-1}S^2 \, dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r < 0$  时, 等式成立当且仅当: 余维数为 2, 矩

阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}, r \leq 0$  时, 陈省身类型估计 II 为

$$0 \leq \int_M r(r-1)S^{r-2} |\nabla S|^2 - 4n^2 crS^{r-1}H^2 + 2rS^{r-1} |Dh|^2 \\ + 2ncrS^{r-1}S + n^2 S^r H^2 - 2n^2 rS^{r-1}H^4 - 2rS^{r-1}(2n\rho H^2 + (2 - \frac{1}{p})\rho^2) \, dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r < 0$  时, 等式成立当且仅当: 余维数为 2, 矩

阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}, r \geq 0$  或者

$(M, r) \in T_{2,2}$  时, 李安民类型估计 I 为

$$0 = \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S'H^2 - 3rS^{r-1}S^2) dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅

当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} =$

$0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \geq 0$  或者  $(M, r) \in T_{2,2}$  时, 李安民类型估计 II 为

$$0 \leq \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S'H^2 - 2n^2rS^{r-1}H^4 - 2rS^{r-1}(2npH^2 + \frac{3}{2}p^2) dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅

当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} =$

$0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \leq 0$  时, 李安民类型估计 I 为

$$0 \leq \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S'H^2 - 3rS^{r-1}S^2) dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,

$A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 且  $(M, r) \in T_{1,1}$ ,  $r \leq 0$  时, 李安民类型估计 II 为

$$0 \leq \int_M r(r-1)S^{r-2}|\nabla S|^2 - 4n^2crS^{r-1}H^2 + 2rS^{r-1}|Dh|^2 \\ + 2ncrS^{r-1}S + n^2S^rH^2 - 2n^2rS^{r-1}H^4 - 2rS^{r-1}(2npH^2 + \frac{3}{2}\rho^2) dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,

$A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

## 10.4 $GD_{(n,E)}$ 子流形的 Simons 型积分不等式

对于  $GD_{(n,E)}$  子流形, 利用  $GD_{(n,F)}$  的公式, 我们可以得到 Simons 类型积分不等式。

**定理 10.13:** 假设  $x: M^n \rightarrow N^{n+1}$  为一般流形之中的  $GD_{(n,E)}$  超曲面, 我们有

$$0 = \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 - 2e^S S^2 \\ - 2ne^S h_{ij} \bar{R}_{i(n+1)(n+1)j} H + n^2 e^S H^2 \\ + 2e^S (h_{ij} \bar{R}_{(n+1)ijk, k} - h_{ij} \bar{R}_{(n+1)kki, j}) + 2e^S (h_{ij} h_{kl} \bar{R}_{ijkl} + h_{ij} h_{il} \bar{R}_{jkkil}) dv.$$

**定理 10.14:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n,E)}$  子流形, 我们

有

$$0 = \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 + n^2 e^S H^2 \\ + 2ne^S h_{ij}^{\beta} \bar{R}_{ija}^{\beta} H^{\alpha} + 2e^S (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ + 2e^S (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk})$$

$$\begin{aligned}
& -2e^S(N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) dv \\
& = \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 - 2n^2 e^S H^4 + n^2 e^S H^2 \\
& \quad + 2ne^S h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2e^S (-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
& \quad + 2e^S (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\
& \quad - 2e^S (N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^\alpha H^\beta) dv.
\end{aligned}$$

**定理 10.15:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n, E)}$  子流形, 我们有

- 原流形为一般流形, 余维数大于等于 2 时, 陈省身类型估计 I 为

$$\begin{aligned}
0 \geq & \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 + n^2 e^S H^2 \\
& + 2ne^S h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2e^S (-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
& + 2e^S (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\
& - 2e^S (2 - \frac{1}{p}) S^2 dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为一般流形, 余维数大于等于 2 时, 陈省身类型估计 II 为

$$\begin{aligned}
0 \geq & \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 - 2n^2 e^S H^4 + n^2 e^S H^2 \\
& + 2ne^S h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2e^S (-h_{ij}^\alpha \bar{R}_{jk, k}^\alpha + h_{ij}^\alpha \bar{R}_{kki, j}^\alpha) \\
& + 2e^S (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) \\
& - 2e^S (2npH^2 + (2 - \frac{1}{p})\rho^2) dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 李安民类型估计 I 为

$$\begin{aligned} 0 \geq & \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 + n^2 e^S H^2 \\ & + 2ne^S h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2e^S (-h_{ij}^\alpha \bar{R}_{ijk}^\alpha + h_{ij}^\alpha \bar{R}_{kji}^\alpha) \\ & + 2e^S (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) - 3e^S S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} =$

$S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 李安民类型估计 II 为

$$\begin{aligned} 0 \geq & \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 - 2n^2 e^S H^4 + n^2 e^S H^2 \\ & + 2ne^S h_{ij}^\beta \bar{R}_{j\alpha}^\beta H^\alpha + 2e^S (-h_{ij}^\alpha \bar{R}_{ijk}^\alpha + h_{ij}^\alpha \bar{R}_{kji}^\alpha) \\ & + 2e^S (h_{ij}^\alpha h_{pk}^\alpha \bar{R}_{ipjk} + h_{ij}^\alpha h_{ip}^\alpha \bar{R}_{kpjk} + h_{ij}^\alpha h_{ik}^\beta \bar{R}_{\alpha\beta jk}) - 2e^S (2npH^2 + \frac{3}{2}\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} =$

$S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 10.16:** 假设  $x: M^n \rightarrow R^{n+1}$  为空间形式之中的  $GD_{(n, E)}$  超曲面, 我们有

$$0 = \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 + 2nce^S S - 2e^S S^2 + n^2 e^S H^2 dv.$$

**定理 10.17:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n,E)}$  子流形, 我们有

$$\begin{aligned} 0 &= \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 + 2nce^S S + n^2 e^S H^2 \\ &\quad - 2e^S (N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2) dv \\ &= \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 + 2nce^S S + n^2 e^S H^2 - 2n^2 e^S H^4 \\ &\quad - 2e^S (N(\hat{A}_\alpha \hat{A}_\beta - \hat{A}_\beta \hat{A}_\alpha) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^\alpha H^\beta) dv. \end{aligned}$$

**定理 10.18:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n,E)}$  子流形, 我们有

- 原流形为空间形式, 余维数大于等于 2 时, 陈省身类型估计 I 为

$$0 \geq \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 + 2nce^S S + n^2 e^S H^2 - 2(2 - \frac{1}{p}) e^S S^2 dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于 2 时, 陈省身类型估计 II 为

$$\begin{aligned} 0 &\geq \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 \\ &\quad + 2nce^S S + n^2 e^S H^2 - 2n^2 e^S H^4 - 2e^S (2n\rho H^2 + (2 - \frac{1}{p})\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于 2 时, 李安民类型估计 I 为

$$0 = \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 + 2nce^S S + n^2 e^S H^2 - 3e^S S^2) dv.$$

当  $S = S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 时, 李安民类型估计 II 为

$$0 \leq \int_M e^S |\nabla S|^2 - 4n^2 c e^S H^2 + 2e^S |Dh|^2 + 2nce^S S + n^2 e^S H^2 - 2n^2 e^S H^4 - 2e^S (2npH^2 + \frac{3}{2}p^2) dv.$$

当  $S = S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

## 10.5 $GD_{(n, \log)}$ 子流形的 Simons 型积分不等式

对于  $W_{(n, \log)}$  子流形, 利用上一节的公式, 我们可以得到下面特殊的 Simons 类型积分不等式。

**定理 10.19:** 假设  $x: M^n \rightarrow N^{n+1}$  为一般流形之中的没有测地点的  $GD_{(n, \log)}$  超曲面, 我们有

$$0 = \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 - 2S - \frac{2n}{S} h_{ij} \bar{R}_{i(n+1)(n+1)j} H + n^2 \log(S) H^2 + \frac{2}{S} (h_{ij} \bar{R}_{(n+1)ijk, k} - h_{ij} \bar{R}_{(n+1)kki, j}) + 2 \frac{1}{S} (h_{ij} h_{kl} \bar{R}_{ijkl} + h_{ij} h_{il} \bar{R}_{jkil}) dv.$$

**定理 10.20:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的没有测地点的  $GD_{(n, \log)}$  子流形, 我们有

$$0 = \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 + n^2 \log(S) H^2$$



$$\begin{aligned}
& + \frac{2n}{S} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2F'(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + \frac{2}{S} (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\
& - \frac{2}{S} (N(A_{\alpha} \hat{A}_{\beta} - \hat{A}_{\beta} A_{\alpha}) + (S_{\alpha\beta})^2) dv \\
= & \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 - \frac{2n^2}{S} H^4 + n^2 \log(S) H^2 \\
& + \frac{2n}{S} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2 \frac{1}{S} (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + \frac{2}{S} (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\
& - \frac{2}{S} (N(\hat{A}_{\alpha} \hat{A}_{\beta} - \hat{A}_{\beta} \hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n \hat{S}_{\alpha\beta} H^{\alpha} H^{\beta}) dv.
\end{aligned}$$

**定理 10.21:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的没有测地点的  $GD_{(n, F)}$  子流形, 我们有

- 原流形为一般流形, 余维数大于等于 2 时, 陈省身类型估计 I 为

$$\begin{aligned}
0 \geq & \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 + n^2 \log(S) H^2 \\
& + \frac{2n}{S} h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2 \frac{1}{S} (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + \frac{2}{S} (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) - \frac{2}{S} (2 - \frac{1}{p}) S^2 dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \quad \vec{H} = 0, \quad \text{并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为一般流形, 余维数大于等于 2 时, 陈省身类型估计 II 为

$$\begin{aligned}
0 \cong & \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 - \frac{2n^2}{S} H^4 + n^2 \log(S) H^2 \\
& + \frac{2n}{S} h_{ij}^{\beta} \bar{R}_{i\alpha}^{\beta} H^{\alpha} + 2 \frac{1}{S} (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + \frac{2}{S} (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\
& = \frac{2}{S} (2n\phi H^2 + (2 - \frac{1}{p})\rho^2) dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 李安民类型估计 I 为

$$\begin{aligned}
0 \cong & \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 + n^2 \log(S) H^2 \\
& + \frac{2n}{S} h_{ij}^{\beta} \bar{R}_{i\alpha}^{\beta} H^{\alpha} + 2 \frac{1}{S} (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + \frac{2}{S} (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) - 3S dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} =$

$$S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 时, 李安民类型估计 II 为

$$\begin{aligned}
0 \cong & \int_M \frac{-1}{S^2} |\nabla S|^2 + \frac{2}{S} |Dh|^2 - \frac{2n^2}{S} H^4 + n^2 \log(S) H^2 \\
& + \frac{2n}{S} h_{ij}^{\beta} \bar{R}_{i\alpha}^{\beta} H^{\alpha} + 2 \frac{1}{S} (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha})
\end{aligned}$$

$$+ \frac{2}{S} (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpij} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) - \frac{2}{S} (2n\rho H^2 + \frac{3}{2}\rho^2) dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = 0 = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 10.22:** 假设  $x: M^n \rightarrow R^{n+1}$  为空间形式之中的没有测地点的  $GD_{(n, F)}$  超曲面, 我们有

$$0 = \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 + 2nc - 2S + n^2 \log(S) H^2 dv.$$

**定理 10.23:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的没有测地点的  $GD_{(n, F)}$  子流形, 我们有

$$\begin{aligned} 0 &= \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 + 2nc + n^2 \log(S) H^2 \\ &\quad - \frac{2}{S} (N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) + (S_{\alpha\beta})^2) dv \\ &= \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 \\ &\quad + 2nc + n^2 \log(S) H^2 - \frac{2n^2}{S} H^4 \\ &\quad - \frac{2}{S} (N(\hat{A}_{\alpha} \hat{A}_{\beta} - \hat{A}_{\beta} \hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^{\alpha} H^{\beta}) dv. \end{aligned}$$

**定理 10.24:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中没有测地点的  $GD_{(n, F)}$  子流形, 我们有

• 原流形为空间形式, 余维数大于等于 2 时, 陈省身类型估计 I 为

$$0 \geq \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 + 2nc + n^2 \log(S) H^2 - 2(2 - \frac{1}{p}) S dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于2时, 陈省身类型估计 II 为

$$0 \geq \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 \\ + 2nc + n^2 \log(S) H^2 - \frac{2n^2}{S} H^4 - \frac{2}{S} (2n\rho H^2 + (2 - \frac{1}{p})\rho^2) \, dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于2时, 李安民类型估计 I 为

$$0 = \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 + 2nc + n^2 \log(S) H^2 - 3S \, dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} =$

$$S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 原流形为空间形式, 余维数大于等于2时, 李安民类型估计 II 为

$$0 \leq \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2 c}{S} H^2 + \frac{2}{S} |Dh|^2 \\ + 2nc + n^2 \log(S) H^2 - \frac{2n^2}{S} H^4 - \frac{2}{S} (2n\rho H^2 + \frac{3}{2}\rho^2) \, dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} =$

$S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

## 10.6 $GD_{(n, \sin)}$ 子流形的 Simons 型积分不等式

对于  $GD_{(n, \sin)}$  子流形, 利用  $GD_{(n, F)}$  子流形的公式, 我们可以得到下面特殊的 Simons 类型积分不等式.

**定理 10.25:** 假设  $x: M^n \rightarrow N^{n+1}$  为一般流形之中的  $GD_{(n, \sin)}$  超曲面, 我们有

$$\begin{aligned} 0 = & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 - 2\cos(S) S^2 \\ & - 2n\cos(S) h_{ij} \bar{R}_{i(n+1)(n+1)j} H + n^2 \sin(S) H^2 \\ & + 2\cos(S) (h_{ij} \bar{R}_{(n+1)ijk, k} - h_{ij} \bar{R}_{(n+1)kki, j}) + 2F'(S) (h_{ij} h_{kl} \bar{R}_{ijk} + h_{ij} h_{il} \bar{R}_{jkli}) dv. \end{aligned}$$

**定理 10.26:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n, \sin)}$  子流形, 我们有

$$\begin{aligned} 0 = & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 + n^2 \sin(S) H^2 \\ & + 2n\cos(S) h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2\cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\ & - 2\cos(S) (N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) + (S_{\alpha\beta})^2) dv \\ = & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 - 2n^2 \cos(S) H^4 + n^2 \sin(S) H^2 \\ & + 2n\cos(S) h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2F'(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\ & - 2\cos(S) (N(\hat{A}_{\alpha} \hat{A}_{\beta} - \hat{A}_{\beta} \hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^{\alpha} H^{\beta}) dv. \end{aligned}$$

**定理 10.27:** 假设  $x: M^n \rightarrow N^{n+p}$ ,  $p \geq 2$  为一般流形之中的  $GD_{(n, \sin)}$  子流形, 我们有

- 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 陈省身类型估计

I 为

$$0 \geq \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 + n^2 \sin(S) H^2$$

$$\begin{aligned}
& + 2n \cos(S) h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2 \cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + 2 \cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\
& - 2 \cos(S) (2 - \frac{1}{p}) S^2 \, dv_i
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 陈省身类型估计 II 为

$$\begin{aligned}
0 \geq & \int_M -\sin(S) |\nabla S|^2 + 2 \cos(S) |Dh|^2 - 2n^2 \cos(S) H^4 + n^2 \sin(S) H^2 \\
& + 2n \cos(S) h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2 \cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + 2 \cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\
& - 2 \cos(S) (2n\rho H^2 + (2 - \frac{1}{p}) \rho^2) \, dv.
\end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 陈省身类型估计 I 为

$$\begin{aligned}
0 \leq & \int_M -\sin(S) |\nabla S|^2 + 2 \cos(S) |Dh|^2 + n^2 \sin(S) H^2 \\
& + 2n \cos(S) h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2 \cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\
& + 2 \cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk})
\end{aligned}$$

$$-2\cos(S)\left(2-\frac{1}{p}\right)S^2 dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 陈省身类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 - 2n^2 \cos(S) H^4 + n^2 \sin(S) H^2 \\ & + 2n \cos(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} + 2\cos(S) (-h_{ij}^{\alpha} \bar{R}_{jk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\ & - 2\cos(S) (2npH^2 + (2 - \frac{1}{p})\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \geq & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 + n^2 \sin(S) H^2 \\ & + 2n \cos(S) h_{ij}^{\beta} \bar{R}_{j\alpha}^{\beta} H^{\alpha} + 2\cos(S) (-h_{ij}^{\alpha} \bar{R}_{jk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\ & - 3\cos(S) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \geq & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 - 2n^2 \cos(S) H^4 + n^2 \sin(S) H^2 \\ & + 2n\cos(S) h_{ij}^{\beta} \bar{R}_{ija}^{\beta} H^{\alpha} + 2\cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\ & - 2\cos(S) (2n\rho H^2 + \frac{3}{2}\rho^2) dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \leq & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 + n^2 \sin(S) H^2 \\ & + 2n\cos(S) h_{ij}^{\beta} \bar{R}_{ija}^{\beta} H^{\alpha} + 2\cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\beta} \bar{R}_{\alpha\beta jk}) \\ & - 3\cos(S) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ ; 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$



• 原流形为一般流形, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 - 2n^2 \cos(S) H^4 + n^2 \sin(S) H^2 \\ & + 2n\cos(S) h_{ij}^{\beta} \bar{R}_{ij\alpha}^{\beta} H^{\alpha} + 2\cos(S) (-h_{ij}^{\alpha} \bar{R}_{ijk, k}^{\alpha} + h_{ij}^{\alpha} \bar{R}_{kki, j}^{\alpha}) \\ & + 2\cos(S) (h_{ij}^{\alpha} h_{pk}^{\alpha} \bar{R}_{ipjk} + h_{ij}^{\alpha} h_{ip}^{\alpha} \bar{R}_{kpjk} + h_{ij}^{\alpha} h_{ik}^{\alpha} \bar{R}_{\alpha\beta jk}) \\ & - 2\cos(S) (2n\rho H^2 + \frac{3}{2}\rho^2) dv, \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 10.28:** 假设  $x: M^n \rightarrow R^{n+1}$  为空间形式之中的  $GD_{(n, \sin)}$  超曲面, 我们有

$$\begin{aligned} 0 = & \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ & + 2nccos(S) S - 2\cos(S) S^2 + n^2 \sin(S) H^2 dv. \end{aligned}$$

**定理 10.29:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n, \sin)}$  子流形, 我们有

$$\begin{aligned} 0 = & \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 + 2nccos(S) S + n^2 \sin(S) H^2 \\ & - 2\cos(S) (N(A_{\alpha} A_{\beta} - A_{\beta} A_{\alpha}) + (S_{\alpha\beta})^2) dv \\ = & \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ & + 2nccos(S) S + n^2 F(S) H^2 - 2n^2 \cos(S) H^4 \\ & - 2\cos(S) (N(\hat{A}_{\alpha} \hat{A}_{\beta} - \hat{A}_{\beta} \hat{A}_{\alpha}) + (\hat{S}_{\alpha\beta})^2 + 2n\hat{S}_{\alpha\beta} H^{\alpha} H^{\beta}) dv. \end{aligned}$$

**定理 10.30:** 假设  $x: M^n \rightarrow R^{n+p}$ ,  $p \geq 2$  为空间形式之中的  $GD_{(n, \sin)}$  子流形, 我们有

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 陈省身类型估计 I 为

$$\begin{aligned} 0 \geq & \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ & + 2nccos(S) S + n^2 \sin(S) H^2 - 2(2 - \frac{1}{p}) \cos(S) S^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 陈省身类型估计 II 为

$$0 \geq \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 + 2nc \cos(S) S \\ + n^2 \sin(S) H^2 - 2n^2 \cos(S) H^4 - 2\cos(S) (2npH^2 + (2 - \frac{1}{p})\rho^2) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 陈省身类型估计 I 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2nc \cos(S) S + n^2 \sin(S) H^2 - 2(2 - \frac{1}{p}) \cos(S) S^2 dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 陈省身类型估计 II 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2 \cos(S) |Dh|^2 \\ + 2nccos(S)S + n^2 \sin(S)H^2 - 2n^2 \cos(S)H^4 - 2 \cos(S) \left(2n\rho H^2 + \left(2 - \frac{1}{p}\right)\rho^2\right) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 李安民类型估计 I 为

$$0 = \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2 \cos(S) |Dh|^2 \\ + 2nccos(S)S + n^2 \sin(S)H^2 - 3 \cos(S)S^2) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \geq 0$  时, 李安民类型估计 II 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2 \cos(S) |Dh|^2 \\ + 2nccos(S)S + n^2 \sin(S)H^2 - 2n^2 \cos(S)H^4 - 2 \cos(S) \left(2n\rho H^2 + \frac{3}{2}\rho^2\right) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 李安民类型估计 I 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2nccos(S)S + n^2 \sin(S)H^2 - 3\cos(S)S^2 \, dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

• 原流形为空间形式, 余维数大于等于 2 且  $\cos(S) \leq 0$  时, 李安民类型估计 II 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 c \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2nccos(S)S + n^2 \sin(S)H^2 - 2n^2 \cos(S)H^4 - 2\cos(S) \left( 2n\rho H^2 + \frac{3}{2}\rho^2 \right) \, dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

## 第 11 章 单位球面中的间隙现象

本章我们依据上一章的 Simons 类积分不等式讨论单位球面之中  $GD_{(n, F)}$  子流形的间隙现象。

### 11.1 $GD_{(n, F)}$ 子流形的间隙现象

对于  $GD_{(n, F)}$  子流形, 利用 Simons 公式, 我们可以讨论间隙现象。

**定理 11.1:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的  $GD_{(n, F)}$  超曲面, 我们有

$$0 = \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 \\ - 2F'(S) S(S-n) \, dv.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M 2F'(S) S(S-n) \, dv = \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) \\ n^2 H^2 \, dv.$$

2. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\int_M 2F'(S) S(S-n) - F''(S) |\nabla S|^2 \, dv = \int_M 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) \\ n^2 H^2 \, dv.$$

3. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M 2F'(S) S(S-n) - (F(S) - 4F'(S)) n^2 H^2 \, dv = \int_M F''(S) |\nabla S|^2 + 2F'(S) \\ |Dh|^2 \, dv.$$

4. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\int_M 2F'(S) S(S-n) - F''(S) |\nabla S|^2 - (F(S) - 4F'(S)) n^2 H^2 \, dv = \int_M + 2F'(S) \\ |Dh|^2 \, dv.$$

5. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M -2F'(S) S(S-n) + F''(S) |\nabla S|^2 + (F(S) - 4F'(S)) n^2 H^2 \, dv$$

$$= \int_M -2F'(S) |Dh|^2 dv.$$

6. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) S(S-n) + (F(S) - 4F'(S)) n^2 H^2 dv \\ &= \int_M -F''(S) |\nabla S|^2 - 2F'(S) |Dh|^2 dv. \end{aligned}$$

7. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) S(S-n) dv + F''(S) |\nabla S|^2 dv \\ &= \int_M -(F(S) - 4F'(S)) n^2 H^2 - 2F'(S) |Dh|^2 dv. \end{aligned}$$

8. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) S(S-n) dv \\ &= \int_M -F''(S) |\nabla S|^2 - (F(S) - 4F'(S)) n^2 H^2 - 2F'(S) |Dh|^2 dv. \end{aligned}$$

9. 当  $S=0$  时, 等式显然成立; 当  $S=n$  时, 等式为

$$0 = \int_M 2F'(n) |Dh|^2 + (F(n) - 4F'(n)) n^2 H^2 dv.$$

**定理 11.2 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的  $GD_{(n, F)}$  超曲面, 且在区间  $[0, n]$  上满足  $F' > 0$ ,  $F - 4F' > 0$ ,  $F'' > 0$  且  $0 \leq S \leq n$  时, 我们有  $S=0$  或者  $S=n$ . 前者为全测地超曲面, 后者为特殊的 Clifford Torus  $C_{(\frac{n}{2}, \frac{n}{2})}$ .

**注释 11.1:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.1 中的第 1 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去。

**定理 11.3:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,  $F'(u) \geq 0$  时, 陈省身类型估计 I 为

$$\begin{aligned} 0 & \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 \\ & \quad - 2(2 - \frac{1}{p}) F'(S) S \left( S - \frac{n}{2 - \frac{1}{p}} \right) dv. \end{aligned}$$

当  $S=S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M 2\left(2 - \frac{1}{p}\right) F'(S) S \left( S - \frac{n}{2 - \frac{1}{p}} \right) dv \\ & \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 dv. \end{aligned}$$

2. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M 2\left(2 - \frac{1}{p}\right) F'(S) S \left( S - \frac{n}{2 - \frac{1}{p}} \right) - F''(S) |\nabla S|^2 dv \\ & \geq \int_M 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 dv. \end{aligned}$$

3. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M 2\left(2 - \frac{1}{p}\right) F'(S) S \left( S - \frac{n}{2 - \frac{1}{p}} \right) - (F(S) - 4F'(S)) n^2 H^2 dv \\ & \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 dv. \end{aligned}$$

4. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M 2\left(2 - \frac{1}{p}\right) F'(S) S \left( S - \frac{n}{2 - \frac{1}{p}} \right) - F''(S) |\nabla S|^2 - (F(S) - 4F'(S)) n^2 H^2 dv \\ & \geq \int_M +2F'(S) |Dh|^2 dv. \end{aligned}$$

5. 当  $S = 0$  时, 不等式变等式; 当  $S = \frac{n}{2 - \frac{1}{p}}$  时, 不等式为

$$0 \geq \int_M 2F' \left( \frac{n}{2 - \frac{1}{p}} \right) |Dh|^2 + \left( F \left( \frac{n}{2 - \frac{1}{p}} \right) - 4F' \left( \frac{n}{2 - \frac{1}{p}} \right) \right) n^2 H^2 dv.$$

**定理 11.4 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的

$GD_{(n, F)}$  子流形, 在区间  $\left[0, \frac{n}{2 - \frac{1}{p}}\right]$  上满足  $F' > 0$ ,  $F - 4F' > 0$ ,  $F'' > 0$  且  $0 \leq S \leq$

$\frac{n}{2 - \frac{1}{p}}$ , 我们有  $S = 0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ . 前者为全测地子流形, 后者为 Veronese

曲面.

**注释 11.2:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.3 中的第 1 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.5:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,  $F'(u) \geq 0$  时, 陈省身类型估计 II 为

$$\begin{aligned} 0 \geq & \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 \\ & - 2\left(2 - \frac{1}{p}\right) F'(S) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) + 2n\left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) \, dv. \end{aligned}$$

当  $S = S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M 2\left(2 - \frac{1}{p}\right) F'(S) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) \, dv \\ & \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 \\ & \quad + 2n\left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) \, dv. \end{aligned}$$

2. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\int_M 2\left(2 - \frac{1}{p}\right) F'(S) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) - F''(S) |\nabla S|^2 \, dv$$



$$\geq \int_M 2F'(S)|Dh|^2 + (F(S) - 4F'(S))n^2H^2 + 2n(1 - \frac{1}{p})F'(S)H^2(2S - nH^2) dv.$$

3. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M 2(2 - \frac{1}{p})F'(S)S(S - \frac{n}{2 - \frac{1}{p}}) - (F(S) - 4F'(S))n^2H^2 dv \\ & \geq \int_M F''(S)|\nabla S|^2 + 2F'(S)|Dh|^2 + 2n(1 - \frac{1}{p})F'(S)H^2(2S - nH^2) dv. \end{aligned}$$

4. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M 2(2 - \frac{1}{p})F'(S)S(S - \frac{n}{2 - \frac{1}{p}}) - F''(S)|\nabla S|^2 - (F(S) - 4F'(S))n^2H^2 dv \\ & \geq \int_M 2F'(S)|Dh|^2 + 2n(1 - \frac{1}{p})F'(S)H^2(2S - nH^2) dv. \end{aligned}$$

5. 当  $S = 0$  时, 不等式变等式; 当  $S = \frac{n}{2 - \frac{1}{p}}$  时, 不等式为

$$\begin{aligned} 0 \geq & \int_M 2F'\left(\frac{n}{2 - \frac{1}{p}}\right)|Dh|^2 + \left(F\left(\frac{n}{2 - \frac{1}{p}}\right) - 4F'\left(\frac{n}{2 - \frac{1}{p}}\right)\right)n^2H^2 \\ & + 2n(1 - \frac{1}{p})F'\left(\frac{n}{2 - \frac{1}{p}}\right)H^2\left(2\frac{n}{2 - \frac{1}{p}} - nH^2\right) dv. \end{aligned}$$

**定理 11.6 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的

$GD_{(n, F)}$  子流形, 在区间  $\left[0, \frac{n}{2 - \frac{1}{p}}\right]$  上满足  $F' > 0$ ,  $F - 4F' > 0$ ,  $F'' > 0$  且  $0 \leq S \leq$

$\frac{n}{2 - \frac{1}{p}}$ , 我们有  $S = 0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ . 前者为全测地子流形, 后者为 Veronese

曲面.

**注释 11.3:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.5 中的第 1 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.7:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,  $F'(u) \leq 0$  时, 陈省身类型估计 I 为

$$0 \leq \int_M F''(S)|\nabla S|^2 + 2F'(S)|Dh|^2 + (F(S) - 4F'(S))n^2H^2$$

$$-2\left(2 - \frac{1}{p}\right)F'(S)S\left(S - \frac{n}{2 - \frac{1}{p}}\right)dv.$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -2\left(2 - \frac{1}{p}\right)F'(S)S\left(S - \frac{n}{2 - \frac{1}{p}}\right) + F''(S)|\nabla S|^2 \\ & + (F(S) - 4F'(S))n^2H^2 dv \geq \int_M -2F'(S)|Dh|^2 dv. \end{aligned}$$

2. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -2\left(2 - \frac{1}{p}\right)F'(S)S\left(S - \frac{n}{2 - \frac{1}{p}}\right) + (F(S) - 4F'(S))n^2H^2 dv \\ & \geq \int_M -F''(S)|\nabla S|^2 - 2F'(S)|Dh|^2 dv. \end{aligned}$$

3. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -2\left(2 - \frac{1}{p}\right)F'(S)S\left(S - \frac{n}{2 - \frac{1}{p}}\right) + F''(S)|\nabla S|^2 dv \\ & \geq \int_M -2F'(S)|Dh|^2 - (F(S) - 4F'(S))n^2H^2 dv. \end{aligned}$$

4. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -2\left(2 - \frac{1}{p}\right)F'(S)S\left(S - \frac{n}{2 - \frac{1}{p}}\right) dv \\ & \geq \int_M -F''(S)|\nabla S|^2 - 2F'(S)|Dh|^2 - (F(S) - 4F'(S))n^2H^2 dv. \end{aligned}$$

5. 当  $S = 0$  时, 不等式变等式; 当  $S = \frac{n}{2 - \frac{1}{p}}$  时, 不等式为

$$0 \leq \int_M 2F' \left( \frac{n}{2 - \frac{1}{p}} \right) |Dh|^2 + \left( F \left( \frac{n}{2 - \frac{1}{p}} \right) - 4F' \left( \frac{n}{2 - \frac{1}{p}} \right) \right) n^2 H^2 dv.$$

**定理 11.8 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形, 在区间  $\left[0, \frac{n}{2 - \frac{1}{p}}\right]$  上满足  $F' < 0$ ,  $F - 4F' < 0$ ,  $F'' < 0$  且  $0 \leq S \leq \frac{n}{2 - \frac{1}{p}}$ , 我们有  $S = 0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ . 前者为全测地子流形, 后者为 Veronese 曲面.

**注释 11.4:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.7 中第 4 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.9:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,  $F'(u) \leq 0$  时, 陈省身类型估计 II 为

$$0 \leq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 - 2F'(S) \left(2 - \frac{1}{p}\right) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) + 2n \left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) dv.$$

当  $S = S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) \left(2 - \frac{1}{p}\right) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) + F''(S) |\nabla S|^2 \\ & + (F(S) - 4F'(S)) n^2 H^2 dv \\ & \geq \int_M -2F'(S) |Dh|^2 - 2n \left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) dv. \end{aligned}$$

2. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) \left(2 - \frac{1}{p}\right) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) + (F(S) - 4F'(S)) n^2 H^2 \, dv \\ & \geq \int_M -F'(S) |\nabla S|^2 - 2F'(S) |Dh|^2 - 2n \left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) \, dv. \end{aligned}$$

3. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) \left(2 - \frac{1}{p}\right) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) + F'(S) |\nabla S|^2 \, dv \\ & \geq \int_M -(F(S) - 4F'(S)) n^2 H^2 - 2F'(S) |Dh|^2 - 2n \left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) \, dv. \end{aligned}$$

4. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -2F'(S) \left(2 - \frac{1}{p}\right) S \left(S - \frac{n}{2 - \frac{1}{p}}\right) \, dv \\ & \geq \int_M -F''(S) |\nabla S|^2 - (F(S) - 4F'(S)) n^2 H^2 - 2F'(S) |Dh|^2 \\ & \quad - 2n \left(1 - \frac{1}{p}\right) F'(S) H^2 (2S - nH^2) \, dv. \end{aligned}$$

5. 当  $S=0$  时, 不等式变等式; 当  $S = \frac{n}{2 - \frac{1}{p}}$  时, 不等式为

$$\begin{aligned} 0 \leq & \int_M 2F' \left( \frac{n}{2 - \frac{1}{p}} \right) |Dh|^2 + \left( F \left( \frac{n}{2 - \frac{1}{p}} \right) - 4F' \left( \frac{n}{2 - \frac{1}{p}} \right) \right) n^2 H^2 \\ & + 2n \left( 1 - \frac{1}{p} \right) F' \left( \frac{n}{2 - \frac{1}{p}} \right) H^2 \left( 2 \frac{n}{2 - \frac{1}{p}} - nH^2 \right) \, dv. \end{aligned}$$

**定理 11.10 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形, 在区间  $[0, \frac{n}{2 - \frac{1}{p}}]$  上满足  $F' < 0$ ,  $F - 4F' < 0$ ,  $F'' < 0$  且  $0 \leq S \leq \frac{n}{2 - \frac{1}{p}}$ , 我们

有  $S=0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ . 前者为全测地子流形, 后者为 Veronese 曲面.

**注释 11.5:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.9 中的第 4 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.11:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,

$F'(u) \geq 0$  时, 李安民类型估计 I 为

$$0 \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 \\ - 3F'(S) S (S - \frac{2n}{3}) dv.$$

当  $S = S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M 3F'(S) S (S - \frac{2n}{3}) dv \\ \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 dv.$$

2. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\int_M 3F'(S) S (S - \frac{2n}{3}) - F''(S) |\nabla S|^2 dv \\ \geq \int_M 2F'(S) |Dh|^2 + (F(S) - 4F'(S)) n^2 H^2 dv.$$

3. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M 3F'(S) S (S - \frac{2n}{3}) - (F(S) - 4F'(S)) n^2 H^2 dv \\ \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 dv.$$

4. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\int_M 3F'(S) S (S - \frac{2n}{3}) - F''(S) |\nabla S|^2 - (F(S) - 4F'(S)) n^2 H^2 dv \\ \geq \int_M 2F'(S) |Dh|^2 dv.$$

5. 当  $S = 0$  时, 不等式变等式; 当  $S = \frac{2n}{3}$  时, 不等式为

$$0 \geq \int_M 2F'(\frac{2n}{3}) |Dh|^2 + (F(\frac{2n}{3}) - 4F'(\frac{2n}{3})) n^2 H^2 dv.$$

**定理 11.12 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形, 在区间  $[0, \frac{2n}{3}]$  上满足  $F' > 0$ ,  $F - 4F' > 0$ ,  $F'' > 0$  且  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S = 0$  或者  $S = \frac{2n}{3}$ . 前者为全测地子流形, 后者为 Veronese 曲面.

**注释 11.6:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.11 中的第 1 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.13:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,  $F'(u) \geq 0$  时, 李安民类型估计 II 为

$$0 \geq \int_M F''(S) |\nabla S|^2 + (F(S) - 4F'(S)) n^2 H^2 + 2F'(S) |Dh|^2 \\ - 3F'(S) S(S - \frac{2n}{3}) + nF'(S) H^2 (2S - nH^2) \, dv.$$

当  $S = S_0 > 0$  且  $F'(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M 3F'(S) S(S - \frac{2n}{3}) \, dv \geq \int_M F''(S) |\nabla S|^2 + (F(S) - 4F'(S)) n^2 H^2 \\ + 2F'(S) |Dh|^2 + nF'(S) H^2 (2S - nH^2) \, dv.$$

2. 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\int_M 3F'(S) S(S - \frac{2n}{3}) - F'(S) |\nabla S|^2 \, dv \\ \geq \int_M (F(S) - 4F'(S)) n^2 H^2 + 2F'(S) |Dh|^2 + nF'(S) H^2 (2S - nH^2) \, dv.$$

3. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\int_M 3F'(S) S(S - \frac{2n}{3}) - (F(S) - 4F'(S)) n^2 H^2 \, dv \\ \geq \int_M F''(S) |\nabla S|^2 + 2F'(S) |Dh|^2 + nF'(S) H^2 (2S - nH^2) \, dv.$$

4. 当  $F' \geq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M 3F'(S)S(S - \frac{2n}{3}) - F''(S)|\nabla S|^2 - (F(S) - 4F'(S))n^2H^2 \, dv \\ & \geq \int_M 2F'(S)|Dh|^2 + nF'(S)H^2(2S - nH^2) \, dv. \end{aligned}$$

5. 当  $S = 0$  时, 不等式变等式; 当  $S = \frac{2n}{3}$  时, 不等式为

$$0 \geq \int_M (F(\frac{2n}{3}) - 4F'(\frac{2n}{3}))n^2H^2 + 2F'(\frac{2n}{3})|Dh|^2 + nF'(\frac{2n}{3})H^2(2\frac{2n}{3} - nH^2) \, dv.$$

**定理 11.14 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形, 在区间  $[0, \frac{2n}{3}]$  上满足  $F' > 0$ ,  $F - 4F' > 0$ ,  $F' > 0$  且  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S = 0$  或者  $S = \frac{2n}{3}$ . 前者为全测地子流形, 后者为 Veronese 曲面.

**注释 11.7:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.13 中的第 1 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.15:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, F)}$  子流形,  $F'(u) \leq 0$  时, 李安民类型估计 I 为

$$\begin{aligned} 0 \leq & \int_M F'(S)|\nabla S|^2 + 2F'(S)|Dh|^2 + (F(S) - 4F'(S))n^2H^2 \\ & - 3F'(S)S(S - \frac{2n}{3}) \, dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) + F''(S)|\nabla S|^2 + (F(S) - 4F'(S))n^2H^2 \, dv \\ & \geq \int_M -2F'(S)|Dh|^2 \, dv. \end{aligned}$$

2. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) + (F(S) - 4F'(S))n^2 H^2 \, dv \\ & \geq \int_M -F'(S)|\nabla S|^2 - 2F'(S)|Dh|^2 \, dv. \end{aligned}$$

3. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) + F'(S)|\nabla S|^2 \, dv \\ & \geq \int_M -2F'(S)|Dh|^2 - (F(S) - 4F'(S))n^2 H^2 \, dv. \end{aligned}$$

4. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) \, dv \\ & \geq \int_M -F'(S)|\nabla S|^2 - 2F'(S)|Dh|^2 - (F(S) - 4F'(S))n^2 H^2 \, dv. \end{aligned}$$

5. 当  $S=0$  时, 不等式变等式; 当  $S = \frac{2n}{3}$  时, 不等式为

$$0 \leq \int_M 2F'(\frac{2n}{3})|Dh|^2 + (F(\frac{2n}{3}) - 4F'(\frac{2n}{3}))n^2 H^2 \, dv.$$

**定理 11.16 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,F)}$  子流形, 在区间  $[0, \frac{2n}{3}]$  上满足  $F' < 0$ ,  $F - 4F' < 0$ ,  $F'' < 0$  且  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S=0$  或者  $S = \frac{2n}{3}$ . 前者为全测地子流形, 后者为 Veronese 曲面.

**注释 11.8:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.15 中的第 4 种情形为例发展得到的, 实际上根据其余的式子附加一些条件同样可以发展间隙定理, 在此略去.

**定理 11.17:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,F)}$  子流形,  $F'(u) \leq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M F''(S)|\nabla S|^2 + (F(S) - 4F'(S))n^2 H^2 + 2F'(S)|Dh|^2 \\ & - 3F'(S)S(S - \frac{2n}{3}) + nF'(S)H^2(2S - nH^2) \, dv. \end{aligned}$$

当  $S=S_0 > 0$  且  $F'(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且



$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

特别的, 根据函数  $F$  及其导数的符号的不同, 我们可以讨论为

1. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) + F'(S)|\nabla S|^2 + (F(S) - 4F'(S))n^2H^2 \, dv \\ & \geq \int_M -2F'(S)|Dh|^2 - nF'(S)H^2(2S - nH^2) \, dv. \end{aligned}$$

2. 当  $F' \leq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) + (F(S) - 4F'(S))n^2H^2 \, dv \\ & \geq \int_M -F'(S)|\nabla S|^2 - 2F'(S)|Dh|^2 - nF'(S)H^2(2S - nH^2) \, dv. \end{aligned}$$

3. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \geq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) + F''(S)|\nabla S|^2 \, dv \\ & \geq \int_M -(F(S) - 4F'(S))n^2H^2 - 2F'(S)|Dh|^2 - nF'(S)H^2(2S - nH^2) \, dv. \end{aligned}$$

4. 当  $F' \leq 0$ ,  $F - 4F' \leq 0$ ,  $F'' \leq 0$  时, 我们有

$$\begin{aligned} & \int_M -3F'(S)S(S - \frac{2n}{3}) \, dv \\ & \geq \int_M -F'(S)|\nabla S|^2 - (F(S) - 4F'(S))n^2H^2 \, dv \\ & \quad - 2F'(S)|Dh|^2 - nF'(S)H^2(2S - nH^2) \, dv. \end{aligned}$$

5. 当  $S = 0$  时, 不等式变等式; 当  $S = \frac{2n}{3}$  时, 不等式为

$$0 \leq \int_M (F(\frac{2n}{3}) - 4F'(\frac{2n}{3}))n^2H^2 + 2F'(\frac{2n}{3})|Dh|^2 + nF'(\frac{2n}{3})H^2(2\frac{2n}{3} - nH^2) \, dv.$$

**定理 11.18 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,F)}$  子流形, 在区间  $[0, \frac{2n}{3}]$  上满足  $F' < 0$ ,  $F - 4F' < 0$ ,  $F'' < 0$  且  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S = 0$  或者  $S = \frac{2n}{3}$ . 前者为全测地子流形, 后者为 Veronese 曲面.

**注释 11.9:** 根据 Simons 积分等式, 我们可以发展很多的间隙定理, 上面的定理是我们以定理 11.17 中的第 4 种情形为例发展得到的, 实际上根据其余的式子

附加一些条件同样可以发展间隙定理, 在此略去。

## 11.2 $GD_{(n,r)}$ 子流形的间隙现象

对于  $GD_{(n,r)}$  子流形, 利用 Simons 公式, 我们可以讨论间隙现象。

**定义 11.1:** 我们定义如下间隙函数为

函数  $g_{(n,1,r,+)}$  为

$$g_{(n,1,r,+)}(t) = \frac{n + \frac{n^2}{2r}t + \sqrt{(n + \frac{n^2}{2r}t)^2 - 8n^2t}}{2}, r \neq 0, t \geq 0;$$

函数  $g_{(n,1,r,-)}$  为

$$g_{(n,1,r,-)}(t) = \frac{n + \frac{n^2}{2r}t - \sqrt{(n + \frac{n^2}{2r}t)^2 - 8n^2t}}{2}, r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,ch,I,+)}$ ,  $p \geq 2$  为

$$g_{(n,p,r,ch,I,+)}(t) = \frac{\frac{n}{2 - \frac{1}{p}} + \frac{n^2}{2r(2 - \frac{1}{p})}t + \sqrt{(\frac{n}{2 - \frac{1}{p}} + \frac{n^2}{2r(2 - \frac{1}{p})}t)^2 - \frac{8n^2}{2 - \frac{1}{p}}t}}{2},$$

$$r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,ch,I,-)}$ ,  $p \geq 2$  为

$$g_{(n,p,r,ch,I,-)}(t) = \frac{\frac{n}{2 - \frac{1}{p}} + \frac{n^2}{2r(2 - \frac{1}{p})}t - \sqrt{(\frac{n}{2 - \frac{1}{p}} + \frac{n^2}{2r(2 - \frac{1}{p})}t)^2 - \frac{8n^2}{2 - \frac{1}{p}}t}}{2},$$

$$r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,ch,II,+)}$ ,  $p \geq 2$  为

$$g_{(n,p,r,ch,II,+)}(t) =$$

$$\frac{\frac{n}{2 - \frac{1}{p}} + \frac{n^2 + 4nr(1 - \frac{1}{p})}{2r(2 - \frac{1}{p})}t + \sqrt{(\frac{n}{2 - \frac{1}{p}} + \frac{n^2 + 4nr(1 - \frac{1}{p})}{2r(2 - \frac{1}{p})}t)^2 - \frac{4n^2(1 - \frac{1}{p})t^2 + 8n^2t}{2 - \frac{1}{p}}}}{2},$$

$$r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,ch,II,-)}$ ,  $p \geq 2$  为

$$g_{(n,p,r,ch,II,-)}(t) =$$

$$\frac{\frac{n}{2-\frac{1}{p}} + \frac{n^2+4nr(1-\frac{1}{p})}{2r(2-\frac{1}{p})}t - \sqrt{(\frac{n}{2-\frac{1}{p}} + \frac{n^2+4nr(1-\frac{1}{p})}{2r(2-\frac{1}{p})}t)^2 - \frac{4n^2(1-\frac{1}{p})t^2+8n^2t}{2-\frac{1}{p}}}}{2},$$

$$r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,li,l,+)} , p \geq 2$  为

$$g_{(n,p,r,li,l,+)}(t) = \frac{\frac{2n}{3} + \frac{n^2}{3r}t + \sqrt{(\frac{2n}{3} + \frac{n^2}{3r}t)^2 - \frac{16n^2}{3}t}}{2}, r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,li,l,-)} , p \geq 2$  为

$$g_{(n,p,r,li,l,-)}(t) = \frac{\frac{2n}{3} + \frac{n^2}{3r}t - \sqrt{(\frac{2n}{3} + \frac{n^2}{3r}t)^2 - \frac{16n^2}{3}t}}{2}, r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,li,II,+)} , p \geq 2$  为

$$g_{(n,p,r,li,II,+)}(t) = \frac{\frac{2n}{3} + \frac{n^2+2nr}{3r}t + \sqrt{(\frac{2n}{3} + \frac{n^2+2nr}{3r}t)^2 - \frac{4n^2t^2+16n^2t}{3}}}{2},$$

$$r \neq 0, t \geq 0;$$

函数  $g_{(n,p,r,li,II,-)} , p \geq 2$  为

$$g_{(n,p,r,li,II,-)}(t) = \frac{\frac{2n}{3} + \frac{n^2+2nr}{3r}t - \sqrt{(\frac{2n}{3} + \frac{n^2+2nr}{3r}t)^2 - \frac{4n^2t^2+16n^2t}{3}}}{2},$$

$$r \neq 0, t \geq 0;$$

**定理 11.19:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的  $GD_{(n,r)}$  超曲面且  $(M, r) \in T_{1,1} \cup T_{2,2}, r \neq 0$ , 我们有

$$\begin{aligned} & \int_M 2rS^{r-1}(S - g_{(n,1,r,+)}(H^2))(S - g_{(n,1,r,-)}(H^2)) dv \\ &= \int_M r(r-1)S^{r-2}|\nabla S|^2 + 2rS^{r-1}|Dh|^2 dv \end{aligned}$$

**定理 11.20 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的极小的  $GD_{(n,r)}$  超曲面且  $(M, r) \in T_{1,1} \cup T_{2,2}, r \geq 1$ , 如果  $0 \leq S \leq n$ , 我们有  $S=0$  或者  $S=n$ 。对于前者是全测地超曲面, 对于后者是特殊的 Clifford Torus  $C_{(\frac{n}{2}, \frac{n}{2})}$ 。

**注释 11.10:** 通过对间隙函数  $g_{(n,1,r,+)}$  更加精细的讨论, 我们可以发展更精密的间隙定理。

**定理 11.21:** 假设  $x: M^n \rightarrow S^{n+p}(1), p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}, r > 0$  或者  $(M, r) \in T_{2,2}$  时, 陈省身类型估计 I 为

$$\begin{aligned} & \int_M 2r(2 - \frac{1}{p}) S^{r-1} (S - g_{(n,p,r, ch, I, +)}(H^2)) (S - g_{(n,p,r, ch, I, -)}(H^2)) \, dv \\ & \geq \int_M r(r-1) S^{r-2} |\nabla S|^2 + 2r S^{r-1} |Dh|^2 \, dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 余

维数为 2, 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.22:** 假设  $x: M^n \rightarrow S^{n+p}(1), p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}, r > 0$  或者  $(M, r) \in T_{2,2}$  时, 陈省身类型估计 II 为

$$\begin{aligned} & \int_M 2r(2 - \frac{1}{p}) S^{r-1} (S - g_{(n,p,r, ch, II, +)}(H^2)) (S - g_{(n,p,r, ch, II, -)}(H^2)) \, dv \\ & \geq \int_M r(r-1) S^{r-2} |\nabla S|^2 + 2r S^{r-1} |Dh|^2 \, dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 余

维数为 2, 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.23 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1), p \geq 2$  为单位球面之中的极小的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}, r \geq 1$  或者  $(M, r) \in T_{2,2}$  时, 如果  $0 \leq S \frac{n}{2 - \frac{1}{p}}$ , 我们

有  $S = 0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ . 对于前者是全测地子流形, 对于后者是 Veronese 曲面。

**定理 11.24:** 假设  $x: M^n \rightarrow S^{n+p}(1), p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}, r < 0$  时, 陈省身类型估计 I 为

$$\int_M -2r(2 - \frac{1}{p}) S^{r-1} (S - g_{(n,p,r, ch, I, +)}(H^2)) (S - g_{(n,p,r, ch, I, -)}(H^2)) \, dv$$

$$\geq \int_M -r(r-1)S^{r-2}|\nabla S|^2 - 2rS^{r-1}|Dh|^2 dv.$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵

$$A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.25:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}, r < 0$  时, 陈省身类型估计 II 为

$$\begin{aligned} & \int_M -2r(2 - \frac{1}{p})S^{r-1}(S - g_{(n,p,r, \text{ch}, l, +)}(H^2))(S - g_{(n,p,r, \text{ch}, l, -)}(H^2)) dv \\ & \geq \int_M -r(r-1)S^{r-2}|\nabla S|^2 - 2rS^{r-1}|Dh|^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵

$$A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.26:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}, r > 0$  或者  $(M, r) \in T_{2,2}$  时, 李安民类型估计 I 为

$$\begin{aligned} & \int_M 3rS^{r-1}(S - g_{(n,p,r, li, l, +)}(H^2))(S - g_{(n,p,r, li, l, -)}(H^2)) dv \\ & \geq \int_M r(r-1)S^{r-2}|\nabla S|^2 + 2rS^{r-1}|Dh|^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}, r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当:

$$\text{矩阵 } A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0,$$

$$\vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.27:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 李安民类型估计 II 为

$$\begin{aligned} & \int_M 3rS^{r-1} (S - g_{(n,p,r,li,l,+)}(H^2))(S - g_{(n,p,r,li,l,-)}(H^2)) \, dv \\ & \geq \int_M r(r-1)S^{r-2} |\nabla S|^2 + 2rS^{r-1} |Dh|^2 \, dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r > 0$  或者  $(M, r) \in T_{2,2}$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.28 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的最小的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}$ ,  $r \geq 1$  或者  $(M, r) \in T_{2,2}$  时, 如果  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S = 0$  或者  $S = \frac{2n}{3}$ . 对于前者是全测地子流形, 对于后者是 Veronese 曲面。

**定理 11.29:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 李安民类型估计 I 为

$$\begin{aligned} & \int_M -3rS^{r-1} (S - g_{(n,p,r,li,l,+)}(H^2))(S - g_{(n,p,r,li,l,-)}(H^2)) \, dv \\ & \geq \int_M -r(r-1)S^{r-2} |\nabla S|^2 - 2rS^{r-1} |Dh|^2 \, dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.30:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,r)}$  子流形且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 李安民类型估计 II 为

$$\begin{aligned} & \int_M -3rS^{r-1}(S - g_{(n,p,r,li,H,+)}(H^2))(S - g_{(n,p,r,li,H,-)}(H^2)) dv \\ & \geq \int_M -r(r-1)S^{r-2}|\nabla S|^2 - 2rS^{r-1}|Dh|^2 dv. \end{aligned}$$

当  $S \equiv S_0 > 0$  且  $(M, r) \in T_{1,1}$ ,  $r < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,

$A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

### 11.3 $GD_{(n,E)}$ 子流形的间隙现象

对于  $GD_{(n,E)}$  子流形, 利用 Simons 公式, 我们可以讨论间隙现象。

**定义 11.2:** 我们定义一些间隙函数

函数  $g_{(n,1,E,+)}$  为

$$g_{(n,1,E,+)}(t) = \frac{n + \sqrt{n^2 - 6n^2t}}{2};$$

函数  $g_{(n,1,E,-)}$  为

$$g_{(n,1,E,-)}(t) = \frac{n - \sqrt{n^2 - 6n^2t}}{2};$$

函数  $g_{(n,p,E,ch,l,+)}$ ,  $p \geq 2$  为

$$g_{(n,p,E,ch,l,+)}(t) = \frac{\frac{n}{2 - \frac{1}{p}} + \sqrt{\left(\frac{n}{2 - \frac{1}{p}}\right)^2 - \frac{6n^2t}{2 - \frac{1}{p}}}}{2}, t \geq 0;$$

函数  $g_{(n,p,E,ch,l,-)}$ ,  $p \geq 2$  为

$$g_{(n,p,E,cl,I,-)}(t) = \frac{\frac{n}{2-\frac{1}{p}} - \sqrt{\left(\frac{n}{2-\frac{1}{p}}\right)^2 - \frac{6n^2t}{2-\frac{1}{p}}}}{2}, t \geq 0;$$

函数  $g_{(n,p,E,cl,II,+)} , p \geq 2$  为

$$g_{(n,p,E,cl,II,+)}(t) = \frac{\frac{n}{2-\frac{1}{p}} + \frac{2n(1-\frac{1}{p})}{2-\frac{1}{p}}t + \sqrt{\left(\frac{n}{2-\frac{1}{p}} + \frac{2n(1-\frac{1}{p})}{2-\frac{1}{p}}t\right)^2 - \frac{4n^2(1-\frac{1}{p})t^2 + 6n^2t}{2-\frac{1}{p}}}}{2},$$

$t \geq 0;$

函数  $g_{(n,p,E,cl,II,-)} , p \geq 2$  为

$$g_{(n,p,E,cl,II,-)}(t) = \frac{\frac{n}{2-\frac{1}{p}} + \frac{2n(1-\frac{1}{p})}{2-\frac{1}{p}}t - \sqrt{\left(\frac{n}{2-\frac{1}{p}} + \frac{2n(1-\frac{1}{p})}{2-\frac{1}{p}}t\right)^2 - \frac{4n^2(1-\frac{1}{p})t^2 + 6n^2t}{2-\frac{1}{p}}}}{2},$$

$t \geq 0;$

函数  $g_{(n,p,E,li,I,+)} , p \geq 2$  为

$$g_{(n,p,E,li,I,+)}(t) = \frac{\frac{2n}{3} + \sqrt{\left(\frac{2n}{3}\right)^2 - 4n^2t}}{2}, t \geq 0;$$

函数  $g_{(n,p,E,li,I,-)} , p \geq 2$  为

$$g_{(n,p,E,li,I,-)}(t) = \frac{\frac{2n}{3} - \sqrt{\left(\frac{2n}{3}\right)^2 - 4n^2t}}{2}, t \geq 0;$$

函数  $g_{(n,p,E,li,II,+)} , p \geq 2$  为

$$g_{(n,p,E,li,II,+)}(t) = \frac{\frac{2n}{3} + \frac{2n}{3}t + \sqrt{\left(\frac{2n}{3} + \frac{2n}{3}t\right)^2 - \frac{4n^2t^2 + 12n^2t}{3}}}{2}, t \geq 0;$$

函数  $g_{(n,p,E,li,II,-)} , p \geq 2$  为

$$g_{(n,p,E,li,II,-)}(t) = \frac{\frac{2n}{3} + \frac{2n}{3}t - \sqrt{\left(\frac{2n}{3} + \frac{2n}{3}t\right)^2 - \frac{4n^2t^2 + 12n^2t}{3}}}{2}, t \geq 0.$$

**定理 11.31:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的  $GD_{(n,E)}$  超曲面, 我们有



$$\int_M 2e^S (S - g_{(n,1,E,+)}(H^2))(S - g_{(n,1,E,-)}(H^2)) = \int_M e^S (|\nabla S|^2 + 2|Dh|^2) dv.$$

**定理 11.32 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的极小  $GD_{(n,E)}$  超曲面, 如果  $0 \leq S \leq n$ , 我们有  $S=0$  或者  $S=n$ 。前者为全测地超曲面, 后者为特殊的 Clifford Torus  $C_{(\frac{n}{2}, \frac{n}{2})}$ 。

**定理 11.33:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,E)}$  子流形, 陈省身类型估计 I 为

$$\begin{aligned} & \int_M 2(2 - \frac{1}{p}) e^S (S - g_{(n,p,E,Ch,I,+)}(H^2))(S - g_{(n,p,E,Ch,I,-)}(H^2)) dv \\ & \geq \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 dv \end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.34:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n,E)}$  子流形, 陈省身类型估计 II 为

$$\begin{aligned} & \int_M 2(2 - \frac{1}{p}) e^S (S - g_{(n,p,E,Ch,II,+)}(H^2))(S - g_{(n,p,E,Ch,II,-)}(H^2)) dv \\ & \geq \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 dv \end{aligned}$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.35 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的极小  $GD_{(n,E)}$  子流形, 如果  $0 \leq S \leq \frac{n}{2 - \frac{1}{p}}$ , 我们有  $S=0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ 。前者为全测地

超曲面, 后者为 Veronese 曲面。

**定理 11.36:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, E)}$  子流形, 李安民类型估计 I 为

$$\begin{aligned} & \int_M 3e^S (S - g_{(n, p, E, li, I, +)}(H^2)) (S - g_{(n, p, E, li, I, -)}(H^2)) \, dv \\ & \geq \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 \, dv \end{aligned}$$

当  $S = S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.37:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, E)}$  子流形, 李安民类型估计 II 为

$$\begin{aligned} & \int_M 3e^S (S - g_{(n, p, E, li, II, +)}(H^2)) (S - g_{(n, p, E, li, II, -)}(H^2)) \, dv \\ & \geq \int_M e^S |\nabla S|^2 + 2e^S |Dh|^2 \, dv \end{aligned}$$

当  $S = S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.38 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的极小  $GD_{(n, E)}$  子流形, 如果  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S = 0$  或者  $S = \frac{2n}{3}$ 。前者为全测地超曲面, 后者为 Veronese 曲面。

**注释 11.11:** 前面的间隙定理附加了极小的条件, 实际上根据间隙函数  $g_{(n, p, E, ch, I, +)}$  等的性质, 可以发展更精细的间隙定理。

## 11.4 $GD_{(n, \log)}$ 子流形的间隙现象

对于  $GD_{(n, \log)}$  子流形, 利用 Simons 公式, 我们可以讨论间隙现象。

**定理 11.39:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的没有测地点的  $GD_{(n, \log)}$  超曲面, 我们有

$$0 = \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2}{S} H^2 + \frac{2}{S} |Dh|^2 + 2n - 2S + n^2 \log(S) H^2 dv.$$

**定理 11.40 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的没有测地点的极小的  $GD_{(n, \log)}$  超曲面, 如果  $0 \leq S \leq n$ , 我们有  $S=0$  或者  $S=n$ 。对于前者是全测地超曲面, 对于后者是特殊的 Clifford Torus  $C_{(\frac{n}{2}, \frac{n}{2})}$ 。

**定理 11.41:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中没有测地点的  $GD_{(n, \log)}$  子流形, 陈省身类型估计 I 为

$$0 \geq \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2}{S} H^2 + \frac{2}{S} |Dh|^2 + 2n + n^2 \log(S) H^2 - 2\left(2 - \frac{1}{p}\right) S dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.42:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中没有测地点的  $GD_{(n, \log)}$  子流形, 陈省身类型估计 II 为

$$0 \geq \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2}{S} H^2 + \frac{2}{S} |Dh|^2 + 2n + n^2 \log(S) H^2 - \frac{2n^2}{S} H^4 \\ - \frac{2}{S} \left( \left(2 - \frac{1}{p}\right) S^2 - 2n \left(1 - \frac{1}{p}\right) S H^2 - \frac{n^2}{p} H^4 \right) dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.43:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的没有测地点的极小的  $GD_{(n, \log)}$  子流形, 如果  $0 \leq S \leq \frac{n}{2 - \frac{1}{p}}$ , 我们有  $S=0$  或者  $S = \frac{n}{2 - \frac{1}{p}}$ . 对于前者

是全测地子流形, 对于后者是 Veronese 曲面.

**定理 11.44:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中没有测地点的  $GD_{(n, \log)}$  子流形, 李安民类型估计 I 为

$$0 = \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2}{S} H^2 + \frac{2}{S} |Dh|^2 + 2n + n^2 \log(S) H^2 - 3S \, dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.45:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中没有测地点的  $GD_{(n, \log)}$  子流形, 李安民类型估计 II 为

$$0 \leq \int_M \frac{-1}{S^2} |\nabla S|^2 - \frac{4n^2}{S} H^2 + \frac{2}{S} |Dh|^2 + 2n + n^2 \log(S) H^2 - \frac{2n^2}{S} H^4 - \frac{2}{S} \left( \frac{3}{2} S^2 - nSH^2 - \frac{n^2}{2} H^4 \right) \, dv.$$

当  $S \equiv S_0 > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = A_{n+4} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.46 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的没有测地点的极小的  $GD_{(n, \log)}$  子流形, 如果  $0 \leq S \leq \frac{2n}{3}$ , 我们有  $S=0$  或者  $S = \frac{2n}{3}$ 。对于前者是全测地子流形, 对于后者是 Veronese 曲面。

**注释 11.12:** 仔细讨论  $GD_{(n, \log)}$  子流形的 Simons 不等式, 可以得到更加精密的间隙定理。

## 11.5 $GD_{(n, \sin)}$ 子流形的间隙现象

对于  $GD_{(n, \sin)}$  子流形, 利用 Simons 公式, 我们可以讨论间隙现象。

**定理 11.47:** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的  $GD_{(n, \sin)}$  超曲面, 我们有

$$0 = \int_M -\sin(S) |\nabla S|^2 + 2\cos(S) |Dh|^2 + (\sin(S) - 4\cos(S)) n^2 H^2 - 2\cos(S) S(S-n) dv.$$

**定理 11.48:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \geq 0$  时, 陈省身类型估计 I 为

$$0 \geq \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 + 2n\cos(S) S + n^2 \sin(S) H^2 - 2\left(2 - \frac{1}{p}\right) \cos(S) S^2 dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,  $S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.49:** 假设  $x: M^n \rightarrow S^{(n+p)}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \geq 0$  时, 陈省身类型估计 II 为

$$0 \geq \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 + 2n\cos(S) S + n^2 \sin(S) H^2 - 2n^2 \cos(S) H^4 - 2\cos(S) \left( \left(2 - \frac{1}{p}\right) S^2 - 2n \left(1 - \frac{1}{p}\right) S H^2 - \frac{n^2}{p} H^4 \right) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.50:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \leq 0$  时, 陈省身类型估计 I 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2n\cos(S) S + n^2 \sin(S) H^2 - 2\left(2 - \frac{1}{p}\right) \cos(S) S^2 dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.51:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \leq 0$  时, 陈省身类型估计 II 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2n\cos(S) S + n^2 \sin(S) H^2 - 2n^2 \cos(S) H^4 \\ - 2\cos(S) \left( \left(2 - \frac{1}{p}\right) S^2 - 2n \left(1 - \frac{1}{p}\right) SH^2 - \frac{n^2}{p} H^4 \right) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 余维数为 2, 矩阵  $A_{n+1} \neq$

$0, A_{n+2} \neq 0, S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, \vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.52:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \geq 0$  时, 李安民类型估计 I 为

$$0 = \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2n\cos(S)S + n^2 \sin(S) H^2 - 3\cos(S) S^2) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.53:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \geq 0$  时, 李安民类型估计 II 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2n\cos(S)S + n^2 \sin(S) H^2 - 2n^2 \cos(S) H^4 \\ - 2\cos(S) \left( \frac{3}{2} S^2 - nSH^2 - \frac{n^2}{2} H^4 \right) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) > 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.54:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \leq 0$  时, 李安民类型估计 I 为

$$0 \leq \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ + 2n\cos(S)S + n^2 \sin(S) H^2 - 3\cos(S) S^2) dv.$$

当  $S \equiv S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}, A_{n+3} = \cdots = A_{n+p} = 0, \vec{H} = 0, \text{ 并且}$$

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

**定理 11.55:** 假设  $x: M^n \rightarrow S^{n+p}(1)$ ,  $p \geq 2$  为单位球面之中的  $GD_{(n, \sin)}$  子流形, 且  $\cos(S) \leq 0$  时, 李安民类型估计 II 为

$$\begin{aligned} 0 \leq & \int_M -\sin(S) |\nabla S|^2 - 4n^2 \cos(S) H^2 + 2\cos(S) |Dh|^2 \\ & + 2n\cos(S) S + n^2 \sin(S) H^2 - 2n^2 \cos(S) H^4 \\ & - 2\cos(S) \left( \frac{3}{2} S^2 - nSH^2 - \frac{n^2}{2} H^4 \right) dv. \end{aligned}$$

当  $S = S_0 > 0$  且  $\cos(S_0) < 0$  时, 等式成立当且仅当: 矩阵  $A_{n+1} \neq 0$ ,  $A_{n+2} \neq 0$ ,

$S_{(n+1)(n+1)} = S_{(n+2)(n+2)} = \frac{S_0}{2}$ ,  $A_{n+3} = \cdots = A_{n+p} = 0$ ,  $\vec{H} = 0$ , 并且

$$A_{n+1} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, A_{n+2} = \frac{\sqrt{S_0}}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

## 11.6 间隙现象的证明

在本节, 我们给出间隙现象的证明。

为了进一步讨论上面的 Simons 不等式的端点对应的超曲面和子流形, 我们需要 Chern - do Carmo - Kobayashi 在他们的著名的文章之中<sup>[3]</sup>提出的两个重要结论, 其中一个为引理, 另一个被称为主定理。为了表述方便, 我们采用一些记号。对于一个超曲面, 我们用

$$h_{ij} = h_{ij}^{n+1}.$$

我们选择局部正交标架, 使得

$$h_{ij} = 0, \quad \forall i \neq j,$$

并且假设

$$h_i = h_{ii}.$$

**引理 11.1** (参见[2, 3]): 假设  $x: M^n \rightarrow S^{n+1}(1)$  是单位球面之中的紧致无边超曲面并且满足  $\nabla h = 0$ , 那么我们有两种情形。



**情形 1:**  $h_1 = \cdots = h_n = \lambda = \text{constant}$ , 并且  $M$  要么是全脐 ( $\lambda > 0$ ) 超曲面要么是全测地 ( $\lambda = 0$ ) 超曲面;

**情形 2:**  $h_1 = \cdots = h_m = \lambda = \text{constant} > 0$ ,  $h_{m+1} = \cdots = h_n = -\frac{1}{\lambda}$ ,  $1 \leq m \leq n-1$ , 并且  $M$  是两个子流形的黎曼乘积  $M_1 \times M_2$ , 此处  $M_1 = S^m(\frac{1}{\sqrt{1+\lambda^2}})$ ,  $M_2 = S^{n-m}(\frac{\lambda}{\sqrt{1+\lambda^2}})$ 。不是一般性, 我们可以假设  $\lambda > 0$  并且  $1 \leq m \leq \frac{n}{2}$ 。

**引理 11.2** (参见文献[2, 3]): Clifford torus  $C_{m, n-m}$  和 Veronese 曲面是单位球面  $S^{n+p}(1)$  之中的唯一的满足  $S = \frac{n}{2 - \frac{1}{p}}$  的极小子流形 ( $H=0$ )。

本章发展了很多间隙定理, 其证明的思路完全一样。下面的间隙定理是定理 11.2, 只需证明它即可, 其余的间隙定理同理可证。

**定理 11.56 (间隙定理):** 假设  $x: M^n \rightarrow S^{n+1}(1)$  为单位球面之中的  $GD_{(n, F)}$  超曲面, 且在区间  $[0, n]$  上满足  $F' > 0$ ,  $F - 4F' > 0$ ,  $F'' > 0$  且  $0 \leq S \leq n$  时, 我们有  $S = 0$  或者  $S = n$ 。前者为全测地超曲面, 后者为特殊的 Clifford Torus  $C_{(\frac{n}{2}, \frac{n}{2})}$ 。

**证明:** 当  $F' \geq 0$ ,  $F - 4F' \geq 0$ ,  $F'' \geq 0$  时, 有

$$\begin{aligned} LHS &= \int_M 2F'(S)S(S-n) \, dv \\ &= \int_M F''(S)|\nabla S|^2 + 2F'(S)|Dh|^2 + (F(S) - 4F'(S))n^2 H^2 \, dv = RHS. \end{aligned}$$

因此, 当  $0 \leq S \leq n$  时, 有估计

$$LHS \leq 0, \quad RHS \geq 0$$

又因为

$$LHS = RHS$$

所以

$$LHS = 0, \quad RHS = 0$$

由此推出

$$S = 0 \text{ 或者 } S = n, \quad Dh = 0, \quad H = 0$$

对于前者为全测地超曲面, 对于后者由上面的两个引理知为 Clifford Torus, 根据第 8 章的例子, 可知为特殊的  $C_{(\frac{n}{2}, \frac{n}{2})}$ 。

**注释 11.13:** 实际上, 对于高余维情形的间隙定理, 由陈省身估计和李安民估计立即可得。

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[General Information]

ON THE GAP PHENOMENON OF CURVATURE  
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- 7□ □ □ □ □ □
  - 7.1  $\mathcal{G}$ □ n, F□ □ □ □ □ □ □ □ □
  - 7.2  $\mathcal{G}$ □ n, r□ □ □ □ □ □ □ □
  - 7.3  $\mathcal{G}$ □ n, E□ □ □ □ □ □ □ □
  - 7.4  $\mathcal{G}$ □ n, log□ □ □ □ □ □ □ □
  - 7.5  $\mathcal{G}$ □ n, sin□ □ □ □ □ □ □ □
- 8□ □ □ □ □ □ □ □ □ □ □
  - 8.1  $\mathcal{G}$ □ n, F□ □ □ □ □ □ □
  - 8.2  $\mathcal{G}$ □ n, r□ □ □ □ □ □ □
  - 8.3  $\mathcal{G}$ □ n, E□ □ □ □ □ □ □
  - 8.4  $\mathcal{G}$ □ n, log□ □ □ □ □ □ □
  - 8.5  $\mathcal{G}$ □ n, sin□ □ □ □ □ □ □
- 9□ □ □ □ □ □ □ □



9.1  $\mathcal{O}(n)$ ,  $F$   $\square \square \square \square \square \square \square \square$

9.2  $\mathcal{O}(n)$ ,  $r$   $\square \square \square \square \square \square \square \square$

9.3  $\mathcal{O}(n)$ ,  $E$   $\square \square \square \square \square \square \square \square$

9.4  $\mathcal{O}(n)$ ,  $\log$   $\square \square \square \square \square \square \square \square$

9.5  $\mathcal{O}(n)$ ,  $\sin$   $\square \square \square \square \square \square \square \square$

9.6  $\square \square \square \square \square \mathcal{O}(n)$ ,  $F$   $\square \square \square \square \square \square \square \square$

$\square$  10  $\square$  Si nons  $\square \square \square \square \square \square$

10.1  $\square \square \square \square \square \square$

10.2  $\mathcal{O}(n)$ ,  $F$   $\square \square \square \square \square$  Si nons  $\square \square \square \square \square \square$

10.3  $\mathcal{O}(n)$ ,  $r$   $\square \square \square \square \square$  Si nons  $\square \square \square \square \square \square$

10.4  $\mathcal{O}(n)$ ,  $E$   $\square \square \square \square \square$  Si nons  $\square \square \square \square \square \square$

10.5  $\mathcal{O}(n)$ ,  $\log$   $\square \square \square \square \square$  Si nons  $\square \square \square \square \square \square$

10.6  $\mathcal{O}(n)$ ,  $\sin$   $\square \square \square \square \square$  Si nons  $\square \square \square \square \square \square$

$\square$  11  $\square \square \square \square \square \square \square \square \square \square$

11.1  $\mathcal{O}(n)$ ,  $F$   $\square \square \square \square \square \square \square \square \square \square$

11.2  $\mathcal{O}(n)$ ,  $r$   $\square \square \square \square \square \square \square \square \square \square$

11.3  $\mathcal{O}(n)$ ,  $E$   $\square \square \square \square \square \square \square \square \square \square$

11.4  $\mathcal{O}(n)$ ,  $\log$   $\square \square \square \square \square \square \square \square \square \square$

11.5  $\mathcal{O}(n)$ ,  $\sin$   $\square \square \square \square \square \square \square \square \square \square$

11.6  $\square \square \square \square \square \square \square \square$

$\square \square \square \square$